DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

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DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

by

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LIST OF SYMBOLS

Symbol	Description	Page
$\mathbf{A}[f]$	Alexander operator	24
$\mathcal{A}_n$	Class of normalized analytic functions f of the form	3
	$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (z \in \mathbb{U})$	
$\mathcal{A} := \mathcal{A}_1$	Class of normalized analytic functions f of the form	2
	$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U})$	
$\mathcal{A}_{n,b}$	Class of all functions $f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$	20
	where b is a fixed non-negative real number.	
$\mathcal{A}_b := \mathcal{A}_{1,b}$	Class of all functions $f(z) = z + bz^2 + a_3z^3 + \cdots$	20
	where b is a fixed non-negative real number.	
$\mathbb{C}$	Complex plane	1
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	12
$\mathcal{CV}_b$	Class of convex functions in $\mathcal{A}_b$	20
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	14
$\mathcal{CV}_b(\alpha)$	Class of convex functions of order α in $\mathcal{A}_b$	20
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	25
D	Domain	1
$\mathcal{D}_n$	$\{\varphi \in \mathcal{H}[1,n] \ : \ \varphi(z) \neq 0, \ z \in \mathbb{U}\}$	54
$\mathcal{D}_{n,-\beta}$	$\{\varphi \in \mathcal{H}_{-\beta}[1,n] \ : \ \varphi(z) \neq 0, \ z \in \mathbb{U}\}$	55
$\mathcal{D} := \mathcal{D}_1$	$\{\varphi \in \mathcal{H}[1,1] \ : \ \varphi(z) \neq 0, \ z \in \mathbb{U}\}$	55
$\mathcal{H}(\mathbb{U})$	Class of analytic functions in $\mathbb{U}$	2
$\mathcal{H}[a,n]$	Class of analytic functions f of the form	2
	$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \ (z \in \mathbb{U})$	
$\mathcal{H}_\beta[a,n]$	Class of analytic functions f with fixed initial coefficient	19
	of the form $p(z) = a + \beta z^n + a_{n+1} z^{n+1} + \cdots, \ (\beta \geq 0, \ z \in \mathbb{U})$	
$\mathcal{HB}(M)[0,1]$	$\{f \in \mathcal{H}[0,1] \ : \ \lvert f(z)\rvert < M, \ M > 0, \ z \in \mathbb{U}\}$	59
$\mathcal{HB}_\beta(M)[0,1]$	$\{f \in \mathcal{H}_\beta[0,1] \ : \ \lvert f(z)\rvert < M, \ M > 0, \ z \in \mathbb{U}\}$	59

$\mathcal{H}_C[0,1]$	$\{f \in \mathcal{H}[0,1] \ : \ f \text{ is convex}, \ z \in \mathbb{U}\}$	63
$\mathcal{H}_{C\beta}[0,1]$	$\{f \in \mathcal{H}_\beta[0,1] \ : \ f \text{ is convex}, \ z \in \mathbb{U}\}$	63
$\mathbf{I}[f]$	Integral operator	24
Im	Imaginary part of a complex number	35
$k(z)$	Koebe function	6
$\mathbf{L}[f]$	Libera operator	25
$\mathbf{L}_\gamma[f]$	Bernardi-Libera-Livingston operator	25
$m(z)$	Möbius function	10
$\mathbb{N}$	Set of all natural numbers	2
$\mathcal{P}_n$	$\{f \in \mathcal{H}[1,n] \ : \ \text{Re } f(z) > 0, \ z \in \mathbb{U}\}$	55
$\mathcal{P}_{n,\beta}$	$\{f \in \mathcal{H}_\beta[1,n] \ : \ \text{Re } f(z) > 0, \ z \in \mathbb{U}\}$	55
Q	Set of analytic and univalent functions on $\overline{\mathbb{U}} \backslash E(q)$	21
$\mathbb{R}$	Set of all real numbers	6
Re	Real part of a complex number	10
$\mathcal{S}$	Class of normalized univalent functions in $\mathcal{A}$	6
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	13
$\mathcal{ST}_b$	Class of starlike functions in $\mathcal{A}_b$	20
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	14
$\mathcal{ST}_b(\alpha)$	Class of starlike functions of order α in $\mathcal{A}_b$	20
$\mathbb{U}$	Open unit disk, $\{z \in \mathbb{C} : \lvert z \rvert < 1\}$	2
$\overline{\mathbb{U}}$	Close unit disk, $\{z \in \mathbb{C} : \lvert z \rvert \leq 1\}$	21
$\partial\mathbb{U}$	Boundary of unit disk, $\{z \in \mathbb{C} : \lvert z \rvert = 1\}$	21
$\psi(r,s,t;z)$	Admissible function	22
$\Psi_n(\Omega,q)$	Class of admissible functions	22
$\Psi_{n,\beta}(\Omega,q)$	Class of β-admissible functions	30
$\Phi_{C,\beta}(\Omega,q)$	Class of β-admissible functions for convexity	79
$\Phi_{C,\beta}(\triangle)$	Class of β-admissible functions for convexity	85
$\Phi_{S,\beta}(\Omega,q)$	Class of β-admissible functions for starlikeness	71
$\Phi_{S,\beta}(\triangle)$	Class of β-admissible functions for starlikeness	78

LIST OF PUBLICATIONS

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SUBORDINASI PEMBEZA FUNGSI ANALISIS DENGAN PEKALI AWAL TETAP

ABSTRAK

Tesis ini mengkaji fungsi analisis bernilai kompleks dalam cakera unit dengan pekali awal tetap atau dengan pekali kedua tetap dalam pengembangan sirinya. Kaedah subordinasi pembeza disesuai dan dipertingkatkan untuk membolehkan penggunaannya, yang diperlukan bagi mendapatkan kelas-kelas fungsi teraku yang sesuai. Tiga masalah penyelidikan dibincangkan di dalam tesis ini. Pertama, subordinasi pembeza linear peringkat kedua

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z) \prec h(z),$$

dipertimbangkan. Syarat-syarat pada fungsi bernilai kompleks A,B,C,D dan h diterbitkan untuk memastikan implikasi pembeza yang bersesuaian diperoleh yang melibatkan penyelesaian p. Untuk pilihan tertentu bagi fungsi h, implikasi-implikasi ini ditafsirkan secara geometri. Hubungkait akan dibuat dengan penemuan-penemuan terdahulu. Hasil subordinasi-subordinasi tersebut seterusnya digunakan untuk mengkaji sifat-sifat rangkuman untuk pengoperasi kamiran linear pada subkelas fungsi analisis dengan pekali awal tetap tertentu. Kepentingannya akan menjadi pengoperasi kamiran linear berbentuk

$$\mathbf{I}[f](z)=\frac{\rho+\gamma}{z^\gamma\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt,$$

dengan ρ dan γ adalah nombor kompleks, dan fungsi ϕ,φ dan f tergolong dalam be-

berapa kelas fungsi analisis. Pengoperasi kamiran linear ditunjukkan memeta subkelas fungsi analisis dengan pekali awal tetap tertentu ke dalam dirinya sendiri. Masalah terakhir yang dipertimbangkan adalah untuk mendapatkan syarat-syarat cukup untuk fungsi analisis dengan pekali awal tetap untuk menjadi bak-bintang atau cembung. Syarat-syarat ini dirangka menggunakan terbitan Schwarzian.

DIFFERENTIAL SUBORDINATION OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

ABSTRACT

This thesis investigates complex-valued analytic functions in the unit disk with fixed initial coefficient or with fixed second coefficient in its series expansion. The methodology of differential subordination is adapted and enhanced to enable its use, which requires obtaining appropriate classes of admissible functions. Three research problems are discussed in this thesis. First, the linear second-order differential subordination

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z),$$

is considered. Conditions on the complex-valued functions A,B,C,D and h are derived to ensure an appropriate differential implication is obtained involving the solutions p. For particular choices of h, these implications are interpreted geometrically. Connections are made with earlier known results. These subordination results are next used to study inclusion properties for linear integral operators on certain subclasses of analytic functions with fixed initial coefficient. Of interest would be the linear integral operator of the form

$$\mathbf{I}[f](z)=\frac{\rho+\gamma}{z^{\gamma}\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt,$$

where ρ and γ are complex numbers, and ϕ, φ and f belong to some classes of analytic functions. The linear integral operator is shown to map certain subclasses of analytic functions with fixed initial coefficient into itself. The final problem considered is in obtaining sufficient conditions for analytic functions with fixed initial coefficient to be starlike or convex. These conditions are framed in terms of the Schwarzian derivative.

CHAPTER 1

INTRODUCTION

The theory of differential subordination is one of the active research topics in the theory of univalent functions. Research on the theory of differential subordination was pioneered by Miller and Mocanu and their monograph [35] compiled a very comprehensive discussion and many applications of the theory. In the last few decades, hundreds of articles related to the subject have been published and many interesting results obtained. By employing the methodology of differential subordination, this thesis investigates the analytic function in unit disk having the fixed initial coefficient in their series expansion.

In the following, a brief introduction of elementary concepts from the theory of univalent functions as well as the theory of differential subordination will be given which will be very useful in later chapters. The relevant definitions, known results and proofs of most of the results can be found in the standard text books by [2, 20, 22, 24, 35].

1.1 Analytic Univalent Functions

Let $\mathbb{C}$ be the complex plane. Let $z_0 \in \mathbb{C}$ and $r > 0$. Denote by

$$\mathrm{D}(z_0, r) := \{z : z \in \mathbb{C}, |z - z_0| < r\}$$

to be the *neighbourhood of* z_0. A set D of $\mathbb{C}$ is called an open set if for every point z_0 in D, there is a neighborhood of z_0 contained in D. An open set D is connected if there is a polygonal path in D joining any pair of points in D.

A *domain* is an open connected set and it is said to be simply connected if the interior domain to every simple closed curve in D lies completely within D. Geometrically, a simply connected domain is a domain without any holes.

A continuous complex-valued function f is *differentiable* at a point $z_0 \in \mathbb{C}$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. Such a function f is said to be *analytic* at z_0 if it is differentiable at z_0 and at every point in some neighbourhood of z_0. It is analytic on D if it is analytic at every point in D. It is known in [2, Corollary 3.3.2, p. 179] that an analytic function f has derivatives of all orders. Thus f has a Taylor series expansion given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!},$$

convergent in some open disk centered at z_0.

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions f of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}).$$

Let $\mathcal{A}$ denote the class of all analytic functions f defined on $\mathbb{U}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each such function f has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}). \tag{1.1}$$

Generally, let $\mathcal{A}_n$ denote the class of all normalized analytic functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \qquad (z \in \mathbb{U},\ n \in \mathbb{N}).$$

where $\mathcal{A}_1 \equiv \mathcal{A}$.

A function f is *univalent* in D if it is one-to-one in D. In other words, the function f does not take the same value twice, that is, $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D with $z_1 \neq z_2$. Thus, a function f is called *locally univalent* at z_0 if it is one-to-one in some neighbourhood of z_0. For an analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0.

Theorem 1.1. *Let f be analytic in a domain* D*. Then f is locally univalent in a neighbourhood of z_0 in* D *if and only if $f'(z_0) \neq 0$.*

Proof. Let f be locally univalent in a neighbourhood of z_0 in D and suppose that $f'(z_0) = 0$. Then

$$g(z) := f(z) - f(z_0)$$

has a zero of order n, $n \geq 2$, at z_0. Since zeroes of a non-constant analytic function are isolated, there exists an $r > 0$ so that both g and f' have no zeroes in the punctured disk $0 < |z - z_0| \leq r$. Let

$$m = \min_{z \in \mathrm{C}} |g(z)|$$

where $\mathrm{C} = \{z : |z - z_0| = r\}$, and $h(z) := f(z_0) - a$, where $a \in \mathbb{C}$ satisfies $0 < |a - f(z_0)| < m$. Then $|h(z)| < |g(z)|$ on C. It follows from Rouche's theorem [20, p. 4] that g and $g + h$ have the same numbers of zeroes inside C. Thus $f(z) - a$ has at least two zeroes inside C. Observe that none of these zeros can be at z_0. Since $f'(z) \neq 0$ in the punctured disk $0 < |z - z_0| \leq r$, these zeros must be simple. Thus $f(z) = a$ at two or more points inside C. This contradicts the assumption that f is locally univalent in a neighbourhood of z_0 in D.

Now, assume that $f'(z_0) \neq 0$ and f is not locally univalent in any neighbourhood of z_0 in D. For each positive integer n, there are points α_n and β_n in $\mathrm{D}(z_0, \rho/n)$ such that $\alpha_n \neq \beta_n$ but $f(\alpha_n) = f(\beta_n)$. Since $\alpha_n, \beta_n \in \mathrm{D}(z_0, \rho/n)$, it follows that

$$\lim_{n\to\infty} \alpha_n = z_0 \qquad \text{and} \qquad \lim_{n\to\infty} \beta_n = z_0.$$

Since $f(\alpha_n) = f(\beta_n)$, by Cauchy's integral formula, it is evident that

$$\begin{aligned} 0 &= \frac{f(\alpha_n) - f(\beta_n)}{\alpha_n - \beta_n} \\ &= \frac{1}{\alpha_n - \beta_n} \left\{ \frac{1}{2\pi i} \int_C \left[\frac{f(z)}{z - \alpha_n} - \frac{f(z)}{z - \beta_n} \right] dz \right\} \\ &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha_n)(z - \beta_n)}\, dz. \end{aligned}$$

Since $\alpha_n \to z_0$ and $\beta_n \to z_0$ as $n \to \infty$, it follows that $f(z)/[(z-\alpha_n)(z-\beta_n)]$ converges uniformly to $f(z)/(z-z_0)^2$. Thus

$$0 = \lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha_n)(z - \beta_n)}\, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2}\, dz = f'(z_0),$$

which contradicts the assumption that $f'(z) \neq 0$. Therefore f must be locally univalent in a neighbourhood of z_0 in D. □

Let γ be a smooth arc represented parametrically by $z = z(t),\ a \leq t \leq b$, and let f be a function defined at all points z on γ. Suppose that γ passes through a point $z_0 = z(t_0),\ a \leq t_0 \leq b$, at which f is analytic and that $f'(z_0) \neq 0$. If $w(t) = f[z(t)]$, then $w'(t_0) = f'[z(t_0)]z'(t_0)$, and this means that

$$\arg w'(t_0) = \arg f'[z(t_0)] + \arg z'(t_0).$$

Let $\psi_0 = \arg w'(t_0)$, $\phi_0 = \arg f'[z(t_0)]$ and $\theta_0 = \arg z'(t_0)$, then $\psi_0 = \phi_0 + \theta_0$. Thus $\phi_0 = \psi_0 - \theta_0$, and the angles ψ_0 and θ_0 differs by the angle of rotation $\phi_0 = \arg f'(z_0)$.

Let γ_1 and γ_2 be two smooth arcs passing through z_0, and let θ_1 and θ_2 be angles of inclination of directed lines tangent to γ_1 and γ_2, respectively, at z_0. Then the quantities $\psi_1 = \phi_0 + \theta_1$ and $\psi_2 = \phi_0 + \theta_2$ are angles of inclination of directed lines tangent to the images curves Γ_1 and Γ_2, respectively, at $w_0 = f(z_0)$. Thus $\psi_2 - \psi_1 = \theta_2 - \theta_1$, that is, the angle $\psi_2 - \psi_1$ from Γ_1 to Γ_2 is the same as the angle $\theta_2 - \theta_1$ from γ_1 to γ_2.

This angle-preserving property leads to the notion of conformal maps. A function that preserves both the magnitude and orientation of angles is said to be *conformal*. The transformation $w = f(z)$ is conformal at z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$. It follows from Theorem 1.1 that the locally univalent functions are also conformal. A function which is both analytic and univalent on D is called a *conformal mapping of* D because of its angle-preserving property.

A *Möbius transformation* is a linear fractional transformation of the form

$$M(z) = \frac{az+b}{cz+d}, \qquad (z \in \overline{\mathbb{C}}),$$

where the coefficients a, b, c, d are complex constants satisfying $ad - bc \neq 0$ and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane. The Möbius transformation M provides a conformal mapping of $\overline{\mathbb{C}}$ onto itself.

The famous Riemann mapping theorem states that any simply connected domain which is not the whole complex plane $\mathbb{C}$, can be mapped conformally onto $\mathbb{U}$.

Theorem 1.2. (Riemann Mapping Theorem) [20, p. 11] *Let* D *be a simply connected domain which is a proper subset of the complex plane* $\mathbb{C}$*. If* ζ *be a given point in* D*, then there is a unique function* f*, analytic and univalent in* D*, which maps* D *conformally onto the unit disk* $\mathbb{U}$ *satisfying* $f(\zeta) = 0$ *and* $f'(\zeta) > 0$*.*

In view of this theorem, the study of conformal mappings on simply connected domains may be restricted to study of analytic univalent functions in $\mathbb{U}$.

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which are univalent and of the form (1.1). Thus $\mathcal{S}$ is the class of all normalized univalent functions in $\mathbb{U}$. An important member of the class $\mathcal{S}$ is the *Koebe function* given by

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n, \qquad (z \in \mathbb{U}), \tag{1.2}$$

which maps $\mathbb{U}$ conformally onto $\mathbb{C} \setminus \{w \in \mathbb{R} : w \leq -1/4\}$. The Koebe function and its rotations $e^{-i\alpha}k(e^{i\alpha}z), \alpha \in \mathbb{R}$, play a very important role in the study of the class $\mathcal{S}$. These functions are extremal for various problems in the class $\mathcal{S}$.

In 1916, Bieberbach [12] conjectured that $|a_n| \leq n, (n \geq 2)$ for f in $\mathcal{S}$. This conjecture is known as Bieberbach conjecture. But he only proved for the case when $n = 2$ and this result was called Bieberbach theorem.

Theorem 1.3. (Bieberbach Theorem) [22, p. 33] *If* $f \in \mathcal{S}$*, then*

$$|a_2| \leq 2.$$

Equality occurs if and only if f *is a rotation of the Koebe function* k*.*

This theorem will be proved later in Section 1.3.

The Bieberbach conjecture was a difficult open problem as many mathematicians have investigated it only for certain values of n. However, in 1985, De Branges [19] successfully proved this conjecture for all coefficients n with the help of the hypergeometric functions.

Theorem 1.4. (Bieberbach Conjecture or de Branges Theorem) [19] *The coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \ldots$. Equality occurs if and only if f is the Koebe function k or one of its rotations.*

The coefficient inequality $|a_2| \leq 2$ from the Bieberbach theorem yields many important properties of univalent functions in the class $\mathcal{S}$. One of the important consequences is the well-known covering theorem due to Koebe.

Theorem 1.5. (Koebe One-Quarter Theorem) [20, p. 31] *The range of every function of the class $\mathcal{S}$ contains the disk $\{w : |w| < 1/4\}$.*

This theorem will be proved later in Section 1.3.

Another important consequence of the Bieberbach theorem is the distortion theorem which provides sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.6. (Distortion Theorem) [20, p. 32] *If $f \in \mathcal{S}$, then*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad (|z| = r < 1).$$

Equality occurs if and only if f is a suitable rotation of the Koebe function k.

This theorem will be proved later in Section 1.3. The distortion theorem can be applied to obtain sharp upper and lower bounds for $|f(z)|$ and that result is known as the growth theorem.

Theorem 1.7. (Growth Theorem) [20, p. 33] *If $f \in \mathcal{S}$, then*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}, \qquad (|z| = r < 1).$$

Equality occurs if and only if f is a suitable rotation of the Koebe function k.

There are many criteria for functions to be univalent. In 1915, Alexander proved an interesting result for the univalence of analytic functions. He showed that, if f is analytic in $\mathbb{U}$ satisfying $\operatorname{Re} f'(z) > 0$ for each $z \in \mathbb{U}$, then f is univalent in $\mathbb{U}$ [22, Theorem 12, p. 88]. Furthermore, in 1935, Noshiro [42] and Warschawski [57] independently proved the following well-known Noshiro-Warschawski theorem.

Theorem 1.8. (Noshiro-Warschawski Theorem) [42, 57] *If an analytic function f satisfies* $\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > 0$ *for some real α and for all z in a convex domain* D, *then f is univalent in* D.

Another criterion for functions to be univalent involved the Schwarzian derivative. The *Schwarzian derivative* of a locally univalent analytic function f in $\mathbb{U}$ is given by

$$\{f,z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Here f' and f'' denote the first and second derivatives of f, respectively. The Schwarzian derivative of any Möbius transformation M is identically zero. Let $\mathscr{S}$ denote the mapping from f to its Schwarzian derivative. It has the property

$$\mathscr{S}(M \circ f) = (\mathscr{S}(M) \circ f) \cdot \left(f'\right)^2 + \mathscr{S}(f) = \mathscr{S}(f),$$

because $\mathscr{S}(M) = 0$ for every Möbius transformation M. This shows that the Schwarzian derivative is invariant under Möbius transformation M [20, p. 259].

In 1949, Nehari [39] discovered that certain estimates on the Schwarzian derivative imply global univalence.

Theorem 1.9. [39, Theorem I, p. 545] *If $f \in \mathcal{S}$, then*

$$|\{f,z\}| \leq \frac{6}{(1-|z|^2)^2}. \tag{1.3}$$

Conversely, if $f \in \mathcal{A}$ satisfies

$$|\{f,z\}| \leq \frac{2}{(1-|z|^2)^2}, \tag{1.4}$$

then f is univalent in $\mathbb{U}$.

The constant 6 and 2 are the best possible. In the same paper, Nehari [39] also obtained the sufficient condition $|\{f,z\}| \leq \pi^2/2$ that implies the univalence of f in $\mathbb{U}$. The constant $\pi^2/2$ is the best possible.

In a similar vein, Pokornyi [50] in 1951 obtained

$$|\{f,z\}| \leq \frac{4}{(1-|z|^2)} \tag{1.5}$$

is sufficient to ensure the univalence of f and the constant 4 is again best possible. Later, Nehari [40] unified all criterion (1.3), (1.4) and (1.5) by establishing the following general criterion of univalence

$$|\{f,z\}| \leq 2p(|z|),$$

where p is a positive continuous even function defined on the interval $(-1,1)$, with the properties that $p(-x) = p(x)$, $(1-x^2)^2 p(x)$ is nonincreasing on the interval $[0,1)$ and the differential equation $y''(x) + p(x)y(x) = 0$ has a solution which does not vanish for $(-1,1)$. The function p is referred as *Nehari function*.

The problem of finding similar bounds on the Schwarzian derivative that would imply univalence was investigated by other authors including Chuaqui et al. [17], Chuaqui et al. [18], Nunokawa et al. [43], Opoolaa and Fadipe-Josepha [44], Ovesea-Tudor and Shigeyoshi [45] and Ozaki and Nunokawa [47].

1.2 Subclasses of Analytic Univalent Functions

This section begins by discussing an important class of functions, so called the functions with positive real part. The class $\mathcal{P}$, consisting of all the functions which have positive real part in $\mathbb{U}$ will be introduced and some of their basic properties will be given as the following.

Definition 1.1. (Functions with Positive Real Part) [22, p. 78] *A normalized analytic function h in* $\mathbb{U}$ *of the form*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad (z \in \mathbb{U}), \tag{1.6}$$

with

$$\text{Re } h(z) > 0$$

is called a function with positive real part in $\mathbb{U}$.

A function with positive real part is also known as a Carathéodory function. An important example of a function of the class $\mathcal{P}$ is the *Möbius function* defined by

$$m(z) \equiv \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n \tag{1.7}$$

which maps $\mathbb{U}$ onto the half-plane $\{\text{Re } w > 0\}$. The role of this Möbius function m is the same as that of Koebe function in the class $\mathcal{S}$. But the function m is not the only extremal functions in the class $\mathcal{P}$, there are many other functions of the form (1.6), which are extremal for the class $\mathcal{P}$.

The following lemma gives the coefficient bound for functions in the class $\mathcal{P}$.

Lemma 1.1. (Carathéodory's Lemma) [20, p. 41] *If $h \in \mathcal{P}$ is of the form (1.6), then the following sharp estimate holds:*

$$|c_n| \leq 2, \quad (n = 1, 2, 3, \ldots).$$

Equality occurs for the Möbius function m.

The following theorem gives the growth and distortion results for the class $\mathcal{P}$.

Theorem 1.10. [24, p. 31] *If $h \in \mathcal{P}$ and $|z| = r < 1$, then*

$$\begin{aligned} \frac{1-r}{1+r} &\leq |h(z)| \leq \frac{1+r}{1-r}, \\ \frac{1-r}{1+r} &\leq \operatorname{Re}\, h(z) \leq \frac{1+r}{1-r}, \\ |h'(z)| &\leq \left(\frac{2}{1-r^2}\right) \operatorname{Re}\, h(z) \leq \frac{2}{(1-r)^2}. \end{aligned}$$

Equalities occurs if and only if h is a suitable rotation of the Möbius function m.

The class $\mathcal{P}$ is directly related to a number of important and basic subclasses of univalent functions. These subclasses include the well-known classes of convex and starlike functions. The geometric properties of these classes along with their relationships with each other will be given.

A set D in $\mathbb{C}$ is called *convex* if for every pair of points w_1 and w_2 lying in D, the line segment joining w_1 and w_2 also lies entirely in D, that is,

$$w_1, w_2 \in \mathrm{D},\ 0 \leq t \leq 1 \qquad \Longrightarrow \qquad tw_1 + (1-t)w_2 \in \mathrm{D}.$$

Definition 1.2. (Convex Functions) [22, p. 107] *If a function $f \in \mathcal{A}$ maps $\mathbb{U}$ onto a convex domain, then f is called a convex function.*

The subclass of $\mathcal{S}$ consisting of all convex functions on $\mathbb{U}$ is denoted by $\mathcal{CV}$. An analytic description of the class $\mathcal{CV}$ is given by the following result.

Theorem 1.11. (Analytical Characterization of Convex Functions) [24, p. 38] *Let $f \in \mathcal{A}$. Then f is convex if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \qquad (z \in \mathbb{U}).$$

For instance, the Möbius function m in (1.7) and the function

$$L(z) = \frac{z}{1-z} \tag{1.8}$$

which maps $\mathbb{U}$ onto the half-plane $\{\operatorname{Re}\ z > -1/2\}$ are convex functions in $\mathbb{U}$. The following theorem gives the coefficient bound for $f \in \mathcal{CV}$ and this result was proved by Loewner [32] in 1917.

Theorem 1.12. [32] *If $f \in \mathcal{CV}$, then*

$$|a_n| \leq 1, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the function L defined in (1.8).

Let w_0 be an interior point of a set D in $\mathbb{C}$. Then D is said to be *starlike with respect to w_0* if the line segment joining w_0 to every other point w in D lies in D, that is,

$$w \in \mathrm{D},\ 0 \leq t \leq 1 \qquad \Longrightarrow \qquad (1-t)w_0 + tw \in \mathrm{D}.$$

For $w_0 = 0$, the set D is called starlike with respect to the origin or simply a starlike domain.

Definition 1.3. (Starlike Functions) [22, p. 108] *If a function f maps $\mathbb{U}$ onto a domain that is starlike with respect to w_0, then f is called a starlike function with respect to w_0. In the special case that $w_0 = 0$, f is simply called a starlike function.*

The subclass of $\mathcal{S}$ consisting of all starlike functions on $\mathbb{U}$ is denoted by $\mathcal{ST}$. An analytic description of the class $\mathcal{ST}$ is given by the following result.

Theorem 1.13. (Analytical Characterization of Starlike Functions) [24, p. 36] *Let $f \in \mathcal{A}$ with $f(0) = 0$. Then f is starlike if and only if $f'(0) \neq 0$ and*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad (z \in \mathbb{U}).$$

The Koebe function in (1.2) is an example of starlike function in $\mathbb{U}$. The following theorem gives the coefficient bound for $f \in \mathcal{ST}$ and this result was proved by Nevanlinna [41] in 1921.

Theorem 1.14. [41] *If $f \in \mathcal{ST}$, then*

$$|a_n| \leq n, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the Koebe function k.

Every convex function is evidently starlike. Thus the subclasses of $\mathcal{S}$ consisting of convex and starlike functions satisfy the following inclusion relation:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{S}.$$

Observe that the classes $\mathcal{CV}$ and $\mathcal{ST}$ are closely related to each other. It is given by the following important relationship:

$$f \in \mathcal{CV} \qquad \Longleftrightarrow \qquad zf'(z) \in \mathcal{ST}, \qquad (z \in \mathbb{U}),$$

due to Alexander [1] in 1915. This result is known as Alexander's theorem.

In 1936, Robertson [53] introduced the classes of convex and starlike functions of order α for $0 \leq \alpha < 1$, which are defined by

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\ \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\ ;\ z \in \mathbb{U} \right\},$$

and

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \mathrm{Re}\ \left(\frac{zf'(z)}{f(z)}\right) > \alpha\ ;\ z \in \mathbb{U} \right\},$$

respectively. In particular, $\mathcal{CV}(0) = \mathcal{CV}$ and $\mathcal{ST}(0) = \mathcal{ST}$. It is clear that

$$\mathcal{CV}(\alpha) \subseteq \mathcal{CV} \qquad \text{and} \qquad \mathcal{ST}(\alpha) \subseteq \mathcal{ST}.$$

Another important relationship between the classes $\mathcal{CV}$ and $\mathcal{ST}$ is given by the classical result of Strohhäcker [55] in 1933. He proved that if $f \in \mathcal{CV}$, then $f \in \mathcal{ST}(1/2)$, where $\mathcal{ST}(1/2)$ is the class of starlike functions of order 1/2. The following theorem is an extension of the result.

Theorem 1.15. [35, p. 115] *If* $0 \le \alpha < 1$, *then the order of starlikeness of convex functions of order* α *is given by*

$$\tau(\alpha) := \tau(\alpha;1,0) = \begin{cases} \frac{2\alpha-1}{2-2^{2(1-\alpha)}}, & if \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\ln 2}, & if \quad \alpha = \frac{1}{2}. \end{cases}$$

The following result gives the growth and distortion theorem for convex functions of order α due to Robertson [53].

Theorem 1.16. [53] *Let* $f \in \mathcal{CV}(\alpha)$, $0 \le \alpha < 1$, *and* $|z| = r < 1$. *Then*

$$\frac{1}{(1+r)^{2(1-\alpha)}} \le |f'(z)| \le \frac{1}{(1-r)^{2(1-\alpha)}}.$$

If $\alpha \neq 1/2$, *then*

$$\frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \le |f(z)| \le \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1},$$

and if $\alpha = 1/2$, *then*

$$Log\,(1+r) \leq |f(z)| \leq -Log\,(1-r).$$

All of these inequalities are sharp. The extremal functions are rotations of

$$f(z) = \begin{cases} \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \frac{1}{2}, \\ -Log\,(1-z), & \alpha = \frac{1}{2}. \end{cases}$$

1.3 Analytic Univalent Functions with Fixed Intial Coefficient

Closely related to the class $\mathcal{S}$ is the class Σ consisting of functions g which are analytic and univalent on $\Delta = \{z \in \mathbb{C} : |z| > 1\}$ except for a simple pole at infinity with residue 1. The Laurent series expansion of such functions is of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \qquad (z \in \Delta). \tag{1.9}$$

This function g maps Δ onto the complement of a connected compact set E. The subclass of Σ that omits $z = 0$ in E is denoted by Σ_0.

Observe that if $f \in \mathcal{S}$ is given by (1.1), then

$$g(z) = \frac{1}{f(1/z)} = z - a_2 + (a_2^2 - a_3)\frac{1}{z} + \cdots \qquad (z \in \Delta),$$

in Σ_0. Conversely, if $g \in \Sigma_0$ is given by (1.9), then

$$f(z) = \frac{1}{g(1/z)} = z - b_0 z^2 + (b_0^2 - b_1)z^3 + \cdots \qquad (z \in \mathbb{U}),$$

in $\mathcal{S}$. In fact, the univalence of f implies the univalence of g as well.

In 1914, Gronwall [26] proved a theorem about the Laurent series coefficients of functions in the class Σ which is known as the area theorem.

Theorem 1.17. (Area Theorem) [26] *If $g \in \Sigma$ is given by (1.9), then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

with equality if and only if $g \in \widetilde{\Sigma}$.

The direct application of the area theorem can be seen clearly in the proof of Bieberbach theorem. Bieberbach theorem states that every function f in the class $\mathcal{S}$ has the property $|a_2| \leq 2$. The following proof can be found in [22, p. 34].

Proof of Theorem 1.3 (Bieberbach Theorem). Suppose that $f \in \mathcal{S}$. A square root transformation yields the function

$$g(z) = \sqrt{f(z^2)} = z + \frac{1}{2}a_2 z^3 + \left(\frac{1}{2}a_3 - \frac{1}{8}a_2^2\right) z^5 + \cdots$$

in $\mathcal{S}$. An inversion to g produce a function

$$h(z) = \frac{1}{g(1/z)} = z - \frac{1}{2}a_2\frac{1}{z} + a_3\frac{1}{z^3} + \cdots$$

in Σ_0. By the area theorem, it follows that

$$\sum_{n=1}^{\infty} n|b_n|^2 = \left|-\frac{a_2}{2}\right|^2 + 3|a_3|^2 + \cdots \leq 1,$$

and so $|-a_2/2|^2 \leq 1$ or $|a_2| \leq 2$, as required. If $a_2 = 2e^{i\alpha}$, for some real α, it is clear that the coefficient $b_n = 0$ for all $n \geq 2$. This implies that h has the form $h(z) = z - e^{i\alpha}/z$. Hence,

$$g(z) = \frac{1}{h(1/z)} = \frac{1}{1/z - e^{i\alpha}z} = \frac{z}{1 - e^{i\alpha}z^2}.$$

Since $f(z^2) = g^2(z) = z^2/(1 - e^{i\alpha}z^2)^2$, and thus $f(z) = z/(1 - e^{i\alpha}z)^2$ is a rotation of the Koebe function. $\square$

The Bieberbach inequality $|a_2| \leq 2$ can be used to prove other properties of functions f in the class $\mathcal{S}$. The famous covering theorem due to Koebe, that is, the Koebe one-quarter theorem is an important application of Bieberbach theorem. It ensures that the image of $\mathbb{U}$ under every f in $\mathcal{S}$ contains an open disk centered at the origin with radius $1/4$. The following proof can be found in [20, p. 31].

Proof of Theorem 1.5 (Koebe One-Quarter Theorem). Every function $f \in \mathcal{S}$ satisfies $|a_2| \leq 2$ by Bieberbach theorem. Suppose that $\omega \notin f(\mathbb{U})$, and the omitted value transformation yields the function

$$g(z) = \frac{\omega f(z)}{\omega - f(z)} = z + \left(a_2 + \frac{1}{\omega}\right) z^2 + \cdots$$

in $\mathcal{S}$. From Bieberbach theorem, it follows that $|a_2 + 1/\omega| \leq 2$ and the triangle inequality yields

$$\left|\frac{1}{\omega}\right| - |a_2| \leq \left|a_2 + \frac{1}{\omega}\right| \leq 2.$$

Since $|a_2| \leq 2$, it is clear that $|1/\omega| \leq 4$, or $|\omega| \geq 1/4$. If $|\omega| = 1/4$, then $|a_2| = 2$, and hence f is some rotation of the Koebe function. $\square$

The proof shows that the Koebe function and its rotations are the only functions in the class $\mathcal{S}$ which omit a value of modulus $1/4$. Thus the range of every other function in $\mathcal{S}$ covers a disk of larger radius.

Bieberbach inequality $|a_2| \leq 2$ also has application to establish the estimate leading to the fundamental theorem about univalent functions, that is, the Koebe distortion theorem. It yields bounds on $|f'(z)|$ as f ranges over the class $\mathcal{S}$. The following proof can be found in [24, p. 15].

Proof of Theorem 1.6 (Distortion Theorem). Suppose that $f \in \mathcal{S}$ and let

$$w(\zeta) = \frac{\zeta + z}{1 + \bar{z}\zeta} = z + (1 - |z|^2)\zeta - \bar{z}(1 - |z|^2)\zeta^2 + \cdots, \qquad (\zeta \in \mathbb{U}),$$

be a Möbius transformation of $\mathbb{U}$ onto $\mathbb{U}$ with $w(0) = z$ and $w'(0) = 1 - |z|^2$. Then the disk automorphism transformation yields the function

$$g(\zeta) = \frac{f(w(\zeta)) - f(z)}{(1 - |z|^2)f'(z)} = \zeta + \left[\frac{(1 - |z|^2)f''(z)}{2f'(z)} - \bar{z}\right]\zeta^2 + \cdots, \qquad (\zeta \in \mathbb{U}),$$

in $\mathcal{S}$. By Bieberbach theorem, it follows that

$$\left|\frac{(1 - |z|^2)f''(z)}{2f'(z)} - \bar{z}\right| \leq 2.$$

Multiplying by $2|z|/(1 - |z|^2)$ to the latter inequality yields

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2}\right| \leq \frac{4|z|}{1 - |z|^2}.$$

Note that the inequality $|\tau| \leq \xi$ implies that $-\xi \leq \mathrm{Re}\,\{\tau\} \leq \xi$. Thus

$$\frac{2|z|^2}{1 - |z|^2} - \frac{4|z|}{1 - |z|^2} \leq \mathrm{Re}\,\left\{\frac{zf''(z)}{f'(z)}\right\} \leq \frac{2|z|^2}{1 - |z|^2} + \frac{4|z|}{1 - |z|^2}. \tag{1.10}$$

Since $f'(z) \neq 0$ and $f'(0) = 1$, there exists an analytic branch of $\log f'$ such that $\log f'(z)|_{z=0} = 0$. For $z = re^{i\theta}$, it follows that

$$\frac{\partial}{\partial r}\log|f'(z)| = \frac{\partial}{\partial r}\mathrm{Re}\,\{\log f'(z)\} = \frac{1}{r}\mathrm{Re}\,\left\{\frac{zf''(z)}{f'(z)}\right\}.$$

It is evident from (1.10) that

$$\frac{2r^2-4r}{1-r^2} \le r\frac{\partial}{\partial r}\log\left|f'\left(re^{i\theta}\right)\right| \le \frac{2r^2+4r}{1-r^2},$$

or

$$\frac{2r-4}{1-r^2} \le \frac{\partial}{\partial r}\log\left|f'\left(re^{i\theta}\right)\right| \le \frac{2r+4}{1-r^2}.$$

Integrating the last inequality with respect to r gives

$$\int_0^r \frac{2u-4}{1-u^2}\,du \le \log\left|f'\left(re^{i\theta}\right)\right| \le \int_0^r \frac{2u+4}{1-u^2}\,du.$$

Since

$$\int_0^r \frac{2u-4}{1-u^2}\,du = \int_0^r \frac{1}{u-1} - \frac{3}{1+u}\,du = \log(1-r) - 3\log(1+r),$$

and

$$\int_0^r \frac{2u+4}{1-u^2}\,du = \int_0^r \frac{1}{1+u} + \frac{3}{1-u}\,du = \log(1+r) - 3\log(1-r),$$

it is clear that

$$\log\left(\frac{1-r}{(1+r)^3}\right) \le \log\left|f'\left(re^{i\theta}\right)\right| \le \log\left(\frac{1+r}{(1-r)^3}\right).$$

Since $\log|f'(0)| = \log 1 = 0$, exponentiating both sides yields

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

□

In view of the influence of the second coefficient on the investigation of geometric properties of the class $\mathcal{S}$, the class of analytic functions with a fixed initial coefficient will be investigated in this thesis. Let $\mathcal{H}_\beta[a,n]$ be the class consisting of all analytic functions f in $\mathbb{U}$ of the form

$$f(z) = a + \beta z^n + a_{n+1}z^{n+1} + \cdots$$

with a fixed initial coefficient β in $\mathbb{C}$. Since its rotation $e^{-i\alpha}f(e^{i\alpha}z)$ is in $\mathcal{H}_\beta[a,n]$, choose α such that $\beta > 0$. In the other words, since $f \in \mathcal{H}_\beta[a,n]$ is rotationally invariance, β is assumed to be a non-negative real number.

Further, let $\mathcal{A}_{n,b}$ be the class consisting of all normalized analytic functions $f \in \mathcal{A}_n$ in $\mathbb{U}$ of the form

$$f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots$$

where the coefficient $a_{n+1} = b$ is a fixed non-negative real number. Write $\mathcal{A}_{1,b}$ as $\mathcal{A}_b$. Thus, the subclass of $\mathcal{A}_b$ consisting of univalent functions is denoted by $\mathcal{S}_b$ and satisfy $\mathcal{S}_b \subset \mathcal{S}$. For $0 \leq \alpha < 1$, let $\mathcal{CV}_b(\alpha)$ and $\mathcal{ST}_b(\alpha)$ be the classes of convex and starlike functions of order α in $\mathcal{S}_b$, respectively. When $\alpha = 0$, these classes are denoted by $\mathcal{CV}_b := \mathcal{CV}_b(0)$ and $\mathcal{ST}_b := \mathcal{ST}_b(0)$.

1.4 Differential Subordination

A differential subordination in the complex plane is a generalization of a differential inequality on the real line. Obtaining information about the properties of a function from its derivatives plays an important role in functions of a real variable. In the study of complex-valued functions, there are differential implications that are characterizing the functions. A simple example is the Noshiro-Warschawski theorem (Theorem 1.8) in Section 1.1.

In the view of the principle of subordination between analytic functions, let f and g be a member of $\mathcal{H}(\mathbb{U})$. Then, the function f is said to be *subordinate* to g in $\mathbb{U}$, written as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z), \qquad (z \in \mathbb{U}),$$

if there exists an analytic function w in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

The basic notations, definitions and theorems stated in this section can be found in the monograph by Miller and Mocanu, which is the main reference that provides a comprehensive discussion on differential subordination. To develop the main idea of Miller and Mocanu's theory on differential subordination, let p be analytic in $\mathbb{U}$ with $p(0) = a$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Let Ω and $\triangle$ be any subsets in $\mathbb{C}$ and consider the differential implication:

$$\left\{\psi\left(p(z), zp'(z), z^2 p''(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega \qquad \Rightarrow \qquad p(\mathbb{U}) \subset \triangle. \tag{1.11}$$

The following definition is required to formulate the fundamental result in the theory of differential subordination.

Definition 1.4. [35, Definition 2.2b, p. 21] *Denote by Q the set of functions q that are analytic and univalent in $\overline{\mathbb{U}} \backslash E(q)$, where*

$$E(q) := \{\zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(q)$.

By the definition of Q, a suitably defined class of functions Ψ as below is a basis to develop the fundamental result in the theory of differential subordination.

Definition 1.5. (Admissibility Condition) [35, Definition 2.3a, p. 27] *Let Ω be a* *domain in* $\mathbb{C}$, $q \in Q$, *and* n *be a positive integer. The class of admissible functions* $\Psi_n(\Omega, q)$ *consists of functions* $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ *satisfying the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega \tag{1.12}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, *and*

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \geq m \,\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

for $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \backslash E(q)$ *and* $m \geq n$. *In particular,* $\Psi_1(\Omega, q) := \Psi(\Omega, q)$.

If $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$, then the admissibility condition (1.12) reduces to

$$\psi\left(q(\zeta), m\zeta q'(\zeta); z\right) \notin \Omega$$

for $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U} \backslash E(q)$ and $m \geq n$.

The next theorem is the fundamental result in the theory of first and second-order differential subordination.

Theorem 1.18. [35, Theorem 2.3b, p. 28] *Let* $\psi \in \Psi_n(\Omega, q)$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

then $p(z) \prec q(z)$.

In view of this theorem, the differential implication of (1.11) is equivalent to

$$\left\{\psi\left(p(z), zp'(z), z^2 p''(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega \qquad \Rightarrow \qquad p(z) \prec q(z),$$

by assuming that $\triangle \neq \mathbb{C}$ is a simply connected domain containing the point a and there is a conformal mapping q of $\mathbb{U}$ onto $\triangle$ satisfying $q(0) = a$.

In the special case when $\Omega \neq \mathbb{C}$ is also a simply connected domain, then $\Omega = h(\mathbb{U})$

where h is a conformal mapping of $\mathbb{U}$ onto Ω such that $h(0) = \psi(a,0,0;0)$. In addition, suppose that the function $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ is analytic in $\mathbb{U}$. In this case, the differential implication of (1.11) is rewritten as

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \qquad \Rightarrow \qquad p(z) \prec q(z).$$

Denote this class by $\Psi_n(h(\mathbb{U}), q)$ or $\Psi_n(h,q)$ and the following result is an immediate consequence of Theorem 1.18.

Theorem 1.19. [35, Theorem 2.3c, p. 30] *Let $\psi \in \Psi_n(h,q)$ with $q(0) = a$. If $p \in \mathcal{H}[a,n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in $\mathbb{U}$, and*

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z),$$

then $p(z) \prec q(z)$.

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.13}$$

then p is called a *solution* of the differential subordination. A univalent function q is called a *dominant of the solution* of the differential subordination if $p(z) \prec q(z)$ for all p satisfying (1.13). A dominant $\tilde{q}$ satisfying $\tilde{q} \prec q$ for all q of (1.13) is said to be the *best dominant* of (1.13). The best dominant is unique up to a rotation of $\mathbb{U}$. If $p(z) \in \mathcal{H}[a,n]$, then $p(z)$ will be called an (a,n)-*solution*, $q(z)$ an (a,n)-*dominant*, and $\tilde{q}(z)$ the *best* (a,n)-*dominant*.

The more general version of (1.13) is given by

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \tag{1.14}$$

where $\Omega \subset \mathbb{C}$ is a simply connected domain containing $h(\mathbb{U})$. Even though $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ may not be analytic in $\mathbb{U}$, the condition in (1.14) shall also be referred as a second-order differential subordination. The same definition of solution, dominant and best dominant as given above can be extended to this generalization.

1.5 Integral Operators

The study of operators plays an important role in geometric function theory. Over the past few decades, many authors have employed various methods to study different types of integral operator $\mathbf{I}$ mapping subsets of $\mathcal{S}$ into $\mathcal{S}$. In this section, some integral operators which map certain subsets $\mathcal{A}$ into $\mathcal{S}$ are given. Noting that an integral operator is sometimes called an integral transformation.

The study of operators can be traced back to 1915 due to Alexander [1]. He introduced an operator $\mathbf{A} : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathbf{A}[f](z) := \int_0^z \frac{f(t)}{t}\, dt,$$

and the operator is now known as Alexander operator. By the Alexander theorem, it is evident that $\mathbf{A}$ is in $\mathcal{CV}$ if and only if $z\mathbf{A}'[f](z) = f(z)$ is in $\mathcal{ST}$.

In 1960, Biernacki [13] conjectured that $f \in \mathcal{S}$ implies $\mathbf{A} \in \mathcal{S}$, but this turned out to be wrong as subsequently, in 1963, Krzyz and Lewandowski [27] disproved it by giving the following counterexample:

$$f(z) = ze^{(i-1)\mathrm{Log}(1-iz)} \equiv \frac{z}{(1-iz)^{1-i}}, \tag{1.15}$$

where Log denotes the principal branch of the logarithm. A function $f \in \mathcal{A}$ is called

α-spirallike and to be univalent if and only if

$$\text{Re}\,\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \qquad |z| < 1,$$

where a real number α lie in the interval $-\pi/2 < \alpha < \pi/2$. Krzyz and Lewandowski showed that f in (1.15) is $(\pi/4)$-spirallike in $\mathbb{U}$, and hence in $\mathcal{S}$, but that the corresponding $\mathbf{A}$ is in fact infinite-valent in $\mathbb{U}$ (see [23, p. 149]).

Meanwhile, Libera [29] and Livingston [31] studied another operator $\mathbf{L} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathbf{L}[f](z) := \frac{2}{z}\int_0^z f(t)\,dt.$$

In 1969, Bernardi [11] generalized this operator by considering the more general operator $\mathbf{L}_\gamma : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathbf{L}_\gamma[f](z) := \frac{1+\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}\,dt, \qquad (\gamma = 1, 2, 3, \ldots). \tag{1.16}$$

Therefore the operator is called as the generalized Bernardi-Libera-Livingston linear operator. By using differential subordination, Pascu [49] and Lewandowski et al. [28] extended the result of the operator in (1.16) to complex values γ for which Re $\gamma \geq 0$, and obtain the following generalization.

Theorem 1.20. [28, 49] *Let* $\mathbf{L}_\gamma$ *be defined by (1.16) and* Re $\gamma \geq 0$. *Then*

(i) $\mathbf{L}_\gamma[\mathcal{ST}] \subset \mathcal{ST}$,

(ii) $\mathbf{L}_\gamma[\mathcal{CV}] \subset \mathcal{CV}$,

(iii) $\mathbf{L}_\gamma[\mathcal{CCV}] \subset \mathcal{CCV}$.

Several authors have investigated the integral operators that have similar properties as the above result. For instance, Singh [54] in 1973 proved that if

$$\mathbf{I}_\rho[f](z) := \left(\frac{\rho+\gamma}{z^\gamma}\int_0^z f^\rho(t)t^{\gamma-1}\,dt\right)^{1/\rho}, \qquad (\rho,\gamma = 1,2,3,\cdots),$$

then $\mathbf{I}_\rho[\mathcal{ST}] \subset \mathcal{ST}$. In 1978, Causey and White [15] showed that if

$$\mathbf{I}[f,g](z) := \left(\frac{\gamma z^\rho}{z^\gamma}\int_0^z \left(\frac{f(t)}{t}\right)^\alpha \left(\frac{g(t)}{t}\right)^\delta t^{\gamma-1}\,dt\right)^{1/\rho}, \qquad (\rho,\gamma = 1,2,3,\cdots),$$

with $\alpha,\delta \geq 0$, then $\mathbf{I}[\mathcal{ST},\mathcal{CV}] \subset \mathcal{ST}$.

The following integral operator investigated by Miller et al. [36] in 1978, has the form

$$\mathbf{I}[f](z) := \left(\frac{\rho+\gamma}{z^\gamma\phi(z)}\int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}\,dt\right)^{1/\rho},$$

where α,ρ,γ and δ are real numbers with certain constraints, ϕ and φ are analytic and f is either starlike or convex. In a later paper in 1991, Miller and Mocanu [34] considered the same integral operator $\mathbf{I}$ but for α,ρ,γ and δ that are complex numbers and f is allowed to be in more general subsets of $\mathcal{A}$. This integral operator becomes the general type of integral operator which maps subsets of $\mathcal{A}$ into $\mathcal{S}$ with suitable restriction on the parameters $\alpha,\rho,\gamma,\delta$ and for f belonging to some classes of analytic functions.

1.6 Scope of the Thesis

This thesis consists of five chapters which include three research problems followed by references. The chapter wise organization of the thesis is as follows.

Chapter 1 presents some elementary concepts of the theory of univalent functions and the theory of differential subordination. All the notations, fundamental definitions and known results required in this thesis are given in this chapter.

Chapter 2 deals with the linear second-order differential subordination of the form

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z),$$

where A,B,C,D and h are complex-valued functions. An appropriate differential implication involving the solutions p is obtained by determining the conditions on the complex-valued functions A,B,C and D. These implications are described geometrically corresponding to a few special cases of h. Under this framework, it gets more computationally involved. Connections are also made with earlier known results.

Chapter 3 focuses on the applications of certain implication results of Chapter 2. The inclusion properties for linear integral operators on certain subclasses of analytic functions with fixed initial coefficient are investigated. These subclasses include functions with positive real part, bounded functions and convex functions. The linear integral operator is derived by using some of the linear differential inequalities from the integral inequalities of Chapter 2.

Chapter 4 studies the subordination of Schwarzian derivative. The differential implication involving Schwarzian derivative is obtained by introducing an appropriate class of admissible functions related to starlikeness or convexity. In particular, sufficient conditions in term of Schwarzian derivative are obtained for functions $f\in\mathcal{A}_b$ to be starlike or convex. Connections are made with previously known results.

In Chapter 5, which is the concluding one, the summary of the work done in this thesis is presented and future prospects of the present work are given.

CHAPTER 2

ADMISSIBLE CLASSES OF ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

2.1 Introduction

The subclass of $\mathcal{H}[0,1]$ consisting of normalized univalent functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in $\mathbb{U}$ is denoted by $\mathcal{S}$. The second coefficient of functions in the class $\mathcal{S}$ plays an important role in the study of geometric properties of the class $\mathcal{S}$. For this reason, there has been continued interest in the investigations of analytic functions with fixed initial coefficient. These include the works of [8, 9, 21, 25, 48] and [56].

The recent work by Ali et al. [7] have extended the established theory of differential subordination pioneered by Miller and Mocanu [35] to functions with pre-assigned second coefficient. These results subsequently have been applied by Ravichandran and Nagpal in [52]. Motivated by these results, this chapter studies further analytic functions with fixed initial coefficient on differential subordination by making modifications and improvements to the works developed in [35, 51].

Recall that $\mathcal{H}_\beta[a,n]$ denote the class consisting of analytic functions p of the form

$$p(z) = a + \beta z^n + p_{n+1}z^{n+1} + \cdots,$$

with the fixed coefficient β is assumed to be a non-negative real number. The following fundamental lemma for functions with fixed initial coefficient is required.

Lemma 2.1. [7, Lemma 2.2, p. 614] *Let $q \in Q$ with $q(0) = a$, and $p \in \mathcal{H}_\beta[a,n]$*

with $p(z) \not\equiv a$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that* $p(z_0) \in q(\partial\mathbb{U})$ *and* $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$, *then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

and

$$\operatorname{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \operatorname{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right),$$

where $q^{-1}(p(z_0)) = \zeta_0 = e^{i\theta_0}$ *and*

$$m \geq n + \frac{|q'(0)| - \beta|z_0|^n}{|q'(0)| + \beta|z_0|^n}.$$

Here Q is the set of functions q that are analytic and univalent in $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) := \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. The subclass of Q for which $q(0) = a$ is denoted by $Q(a)$.

Remark 2.1. *When* $\beta = |q'(0)|$, *the above fundamental lemmas for functions with initial coefficient reduces to the fundamental lemmas introduced by Miller and Mocanu [35, Lemma 2.2d, p. 24]. Though even without this condition, the first two necessary conditions in Lemma 2.1 are the same as the first two necessary conditions in Miller and Mocanu [35, Lemma 2.2d, p. 24] but the class of analytic functions considered is different.*

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ be defined in a domain D and let h be univalent in $\mathbb{U}$. Suppose $p \in \mathcal{H}_\beta[a,n]$, $\left(p(z), zp'(z), z^2p''(z); z\right) \in$ D when $z \in \mathbb{U}$, a nd p satisfies the second-order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{2.1}$$

then p is called a β-*solution* of the second-order differential subordination (2.1). A univalent function q is a β-*dominant* of the second-order differential subordination if $p \prec q$ for all p satisfying the second-order differential subordination (2.1).

The following definition is required in the present investigation.

Definition 2.1. (β-Admissibility Condition) [7, Definition 3.1, p. 616] *Let Ω be a domain in $\mathbb{C}$, $q \in Q$, $\beta \in \mathbb{R}$ with $\beta \leq |q'(0)|$ and n be a positive integer. The class $\Psi_{n,\beta}(\Omega, q)$ consists of functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfying the following conditions:*

(i) $\psi(r,s,t;z)$ *is continuous in a domain* $\mathrm{D} \subset \mathbb{C}^3 \times \mathbb{U}$,

(ii) $(q(0), 0, 0; 0) \in \mathrm{D}$ *and* $\psi(q(0), 0, 0; 0) \in \Omega$,

(iii) $\psi(r_0, s_0, t_0; z_0) \notin \Omega$ *whenever* $(r_0, s_0, t_0; z_0) \in \mathrm{D}$, $r_0 = q(\zeta)$, $s_0 = m\zeta q'(\zeta)$, *and*

$$\operatorname{Re}\left(\frac{t_0}{s_0} + 1\right) \geq m \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

where $|\zeta| = 1$, $q(\zeta)$ *is finite and*

$$m \geq n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}.$$

The class $\Psi_{1,\beta}(\Omega, q)$ *is denoted by* $\Psi_\beta(\Omega, q)$.

If $\beta = |q'(0)|$, then the above concept of β-admissibility coincides with the usual admissibility as Definition 1.5 in Section 1.4. In this case, $\Psi_n(\Omega, q) = \Psi_{n,|q'(0)|}(\Omega, q)$. Further, if $\beta = |q'(0)| = 0$, then it is clearly that $\Psi_{n+1}(\Omega, q) = \Psi_{n,0}(\Omega, q)$. Note that $\Psi_n(\Omega, q) \subset \Psi_{n+1}(\Omega, q)$ [37, Remark 3, p. 159]. Evidently,

$$\Psi_n \equiv \Psi_{n,|q'(0)|} \subseteq \Psi_{n,\beta} \subset \Psi_{n,0} \equiv \Psi_{n+1}.$$

In view of the above inclusions, it is assumed that $0 < \beta \leq |q'(0)|$.

There are two interesting cases for $q(\mathbb{U})$. First, when $q(\mathbb{U})$ is the right half-plane $\triangle = \{w : \text{Re } w > 0\}$. In this case, the function

$$q(z) = \frac{a + \bar{a}z}{1-z}, \qquad z \in \mathbb{U},\ \text{Re } a > 0,$$

is univalent in $\overline{\mathbb{U}} \setminus \{1\}$ and satisfies $q(\mathbb{U}) = \triangle,\ q(0) = a,\ q'(0) = a + \bar{a} = 2\text{Re } a$, and $q \in Q(a)$. Set $\Psi_{n,\beta}(\Omega, a) := \Psi_{n,\beta}(\Omega, q)$, and when $\Omega = \triangle$, denote the class by $\Psi_{n,\beta}\{a\}$. If $a = 1$, then the β-admissibility condition [7, Condition (3.4), p. 619] is as follows:

$$\begin{aligned} &\psi(i\rho, \sigma, \mu + i\nu; z) \notin \Omega \quad \text{whenever} \quad (i\rho, \sigma, \mu + i\nu; z) \in \mathrm{D}, \\ &\sigma \leq -\frac{1}{2}\left(n + \frac{2-\beta}{2+\beta}\right)(1+\rho^2), \quad \text{and} \quad \sigma + \mu \leq 0, \end{aligned} \tag{2.2}$$

where $\rho, \sigma, \mu, \nu \in \mathbb{R}$, and $n \geq 1$.

The following result relates to the case of the right-half plane.

Lemma 2.2. [7, Theorem 3.4, p. 620] *Let* $p \in \mathcal{H}_\beta[a,n]$ *with* Re $a > 0$, $0 < \beta \leq 2\text{Re } a$*.*

(i) Let $\psi \in \Psi_{n,\beta}(\Omega, a)$ *with associated domain* D*. If* $\left(p(z), zp'(z), z^2p''(z); z\right) \in \mathrm{D}$ *and*

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \qquad (z \in \mathbb{U}),$$

then Re $p(z) > 0$.

(ii) Let $\psi \in \Psi_{n,\beta}\{a\}$ *with associated domain* D*. If* $\left(p(z), zp'(z), z^2p''(z); z\right) \in \mathrm{D}$ *and*

$$\text{Re } \psi\left(p(z), zp'(z), z^2p''(z); z\right) > 0, \qquad (z \in \mathbb{U}),$$

then Re $p(z) > 0$.

The second case of interest is when $q(\mathbb{U})$ is the disk $\triangle = \{w : |w| < N\}$. Then the function

$$q(z) = N\frac{Nz+a}{N+\bar{a}z}, \qquad z \in \mathbb{U},\ N>0,\ |a|<N,$$

is univalent in $\overline{\mathbb{U}}$ and satisfies $q(\mathbb{U}) = \triangle,\ q(0)=a,\ q'(0) = (N^2-|a|^2)/N$, and $q \in Q(a)$. In this case, set $\Psi_{n,\beta}(\Omega,N,a) := \Psi_{n,\beta}(\Omega,q)$ and when $\Omega = \triangle$, denote the class by $\Psi_{n,\beta}(N,a)$. If $a=0$, then the β-admissibility condition [7, Condition (3.2), p. 618] is as follows:

$$\begin{gathered}\psi(Ne^{i\theta},Ke^{i\theta},L;z) \notin \Omega \quad \text{whenever} \quad (Ne^{i\theta},Ke^{i\theta},L;z) \in \mathrm{D},\\ K \geq \left(n+\frac{N-\beta}{N+\beta}\right)N, \quad \text{and} \quad \operatorname{Re}\left(Le^{-i\theta}\right) \geq \left(n-\frac{2\beta}{N+\beta}\right)K,\end{gathered} \tag{2.3}$$

where $\theta \in \mathbb{R}$ and $n \geq 1$.

The following result is for the particular case of the disk.

Lemma 2.3. [7, Theorem 3.3(i), p. 619] *Let $p \in \mathcal{H}_\beta[a,n]$ with $N>0,\ |a|<N,\ 0<\beta \leq (N^2-|a|^2)/N$.*

(i) Let $\psi \in \Psi_{n,\beta}(\Omega,N,a)$ with associated domain D. *If $\left(p(z),zp'(z),z^2p''(z);z\right) \in \mathrm{D}$ and*

$$\psi\left(p(z),zp'(z),z^2p''(z);z\right) \in \Omega, \qquad (z \in \mathbb{U}),$$

then $|p(z)|<N$.

(ii) Let $\psi \in \Psi_{n,\beta}(N,a)$ with associated domain D. *If $\left(p(z),zp'(z),z^2p''(z);z\right) \in \mathrm{D}$ and*

$$\left|\psi\left(p(z),zp'(z),z^2p''(z);z\right)\right| < N, \qquad (z \in \mathbb{U}),$$

then $|p(z)|<N$.

2.2 Differential Subordination of Functions with Positive Real Part

Consider the linear second-order differential subordination

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\in\Omega, \tag{2.4}$$

where A,B,C,D are complex-valued functions and Ω is a domain in $\mathbb{C}$.

First let Ω in (2.4) be the right half-plane.

Theorem 2.1. *Let n be a positive integer, $0<\beta\le 2$, and $A(z)=A\ge 0$. Suppose that the functions $B,C,D:\mathbb{U}\to\mathbb{C}$ satisfy* $\operatorname{Re} B(z)\ge A$ *and*

$$\begin{aligned}(\operatorname{Im} C(z))^2 \le &\left[\left(n+\frac{2-\beta}{2+\beta}\right)(\operatorname{Re} B(z)-A)\right]\\ &\times\left[\left(n+\frac{2-\beta}{2+\beta}\right)(\operatorname{Re} B(z)-A)-2\operatorname{Re} D(z)\right].\end{aligned} \tag{2.5}$$

If $p\in\mathcal{H}_\beta[1,n]$ satisfies

$$\operatorname{Re}\left(Az^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\right)>0,$$

then $\operatorname{Re} p(z)>0$.

Proof. Let $\psi(r,s,t;z)=At+B(z)s+C(z)r+D(z)$, Ω be the right half-plane, and $q(z)=(1+z)/(1-z)$. The object is to show that $\psi\in\Psi_{n,\beta}\{1\}$. The function ψ is continuous in the domain $\mathrm{D}=\mathbb{C}^3\times\mathbb{U}$, $(1,0,0;0)\in\mathrm{D}$ and $\operatorname{Re}\psi(1,0,0;0)=\operatorname{Re}(C(0)+D(0))>0$ so that $\psi(1,0,0;0)\in\Omega$. To verify that the β-admissibility condition (2.2) is satisfied, it is enough to show that

$$\operatorname{Re}\psi(i\rho,\sigma,\mu+i\nu;z)\le 0$$

whenever $\sigma\le-\left\{[n+(2-\beta)/(2+\beta)](1+\rho^2)\right\}/2$ and $\sigma+\mu\le 0$, with $\rho,\sigma,\mu,\nu\in\mathbb{R}$ and $n\ge 1$.

Consider

$$\psi(i\rho,\sigma,\mu+i\nu;z)=A(\mu+i\nu)+B(z)\sigma+C(z)i\rho+D(z).$$

Then

$$\begin{aligned}
&\text{Re } \psi(i\rho,\sigma,\mu+i\nu;z)\\
&= \mu A+\sigma \text{Re } B(z)-\rho \text{ Im } C(z)+\text{Re } D(z)\\
&\leq -\sigma A+\sigma \text{Re } B(z)-\rho \text{ Im } C(z)+\text{Re } D(z)\\
&\leq -\frac{1}{2}\left(n+\frac{2-\beta}{2+\beta}\right)(1+\rho^2)(\text{Re } B(z)-A)-\rho \text{ Im } C(z)+\text{Re } D(z)\\
&= -\frac{1}{2}\left(n+\frac{2-\beta}{2+\beta}\right)\rho^2(\text{Re } B(z)-A)-\rho \text{ Im } C(z)\\
&\quad -\frac{1}{2}\left[\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)-2\text{Re } D(z)\right]\\
&= -\frac{1}{2}\frac{\left(n+\frac{2-\beta}{2+\beta}\right)^2\rho^2(\text{Re } B(z)-A)^2}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}-\frac{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)\rho \text{ Im } C(z)}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}\\
&\quad -\frac{1}{2}\frac{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)\left[\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)-2\text{Re } D(z)\right]}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}.
\end{aligned}$$

Using the condition (2.5) to the latter inequality yields

$$\begin{aligned}
&\text{Re } \psi(i\rho,\sigma,\mu+i\nu;z)\\
&\leq -\frac{1}{2}\frac{\left(n+\frac{2-\beta}{2+\beta}\right)^2\rho^2(\text{Re } B(z)-A)^2}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}-\frac{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)\rho \text{ Im } C(z)}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}\\
&\quad -\frac{1}{2}\frac{(\text{Im } C(z))^2}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}\\
&= -\frac{1}{2}\frac{\left[\left(n+\frac{2-\beta}{2+\beta}\right)\rho(\text{Re } B(z)-A)+\text{Im } C(z)\right]^2}{\left(n+\frac{2-\beta}{2+\beta}\right)(\text{Re } B(z)-A)}
\end{aligned}$$

which is non-positive. Hence, $\psi\in\Psi_{n,\beta}\{1\}$ and Lemma 2.2(ii) yield the required result. This completes the proof of the theorem. $\square$

When $\beta = |q'(0)|$, the fact that $\Psi_{n,\beta}(\Omega,q) = \Psi_n(\Omega,q)$ reduces the class $\mathcal{H}_\beta[a,n]$ to the class $\mathcal{H}[a,n]$. In the case $\beta = |q'(0)| = 2$, Theorem 2.1 leads to Theorem 4.1a in Miller and Mocanu [35].

Corollary 2.1. [35, Theorem 4.1a, p. 188] *Let n be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B, C, D : \mathbb{U} \to \mathbb{C}$ satisfy $\text{Re } B(z) \geq A$ and*

$$(\text{Im } C(z))^2 \leq n(\text{Re } B(z) - A) \cdot \text{Re } (nB(z) - nA - 2D(z)).$$

If $p \in \mathcal{H}[1,n]$ satisfies

$$\text{Re } \left(Az^2 p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\right) > 0,$$

then $\text{Re } p(z) > 0$.

Applying Theorem 2.1 to the particular case $A = 0$ and $D(z) = 0$ yields the following first-order result.

Corollary 2.2. *Let n be a positive integer, $0 < \beta \leq 2$, and let B, C be complex-valued functions defined in $\mathbb{U}$, with*

$$|\text{Im } C(z)| \leq \left(n + \frac{2-\beta}{2+\beta}\right) \text{Re } B(z). \tag{2.6}$$

If $p \in \mathcal{H}_\beta[1,n]$ satisfies

$$\text{Re } \left(B(z)zp'(z) + C(z)p(z)\right) > 0,$$

then $\text{Re } p(z) > 0$.

The following example illustrates the similar result of Corollary 2.2.

Example 2.1. *Let $B(z) = \delta$ and $C(z) = 1 - \delta$ with $\delta > 0$. In this case, condition (2.6) in Corollary 2.2 becomes*

$$|\text{Im } (1-\delta)| \leq \left(n + \frac{2-\beta}{2+\beta}\right) \text{Re } \delta.$$

The result follows from Corollary 2.2 as follows:

If $p \in \mathcal{H}_\beta[1,n]$, $0 < \beta \leq 2$, *satisfies*

$$\text{Re }\left(\delta z p'(z) + (1-\delta)p(z)\right) > 0,$$

then Re $p(z) > 0$.

2.3 Differential Subordination of Bounded Functions

In this section, let Ω in (2.4) be the disk of radius $M > 0$ centered at the origin and first consider the case when $A(z) = 0$.

Theorem 2.2. *Let* $M > 0$, $N > 0$, $0 < \beta \leq N$, *and* n *be a positive integer. Suppose* $B, C, D : \mathbb{U} \to \mathbb{C}$ *satisfy* $B(z) \neq 0$,

(i) $\text{Re}\left(\frac{C(z)}{B(z)}\right) \geq -\left(n + \frac{N-\beta}{N+\beta}\right)$, *and*

(ii) $\left|\left(n + \frac{N-\beta}{N+\beta}\right)B(z) + C(z)\right| \geq \frac{1}{N}(M + |D(z)|)$.

If $p \in \mathcal{H}_\beta[0,n]$ *satisfies*

$$\left|B(z)zp'(z) + C(z)p(z) + D(z)\right| < M, \tag{2.7}$$

then $|p(z)| < N$.

Proof. Note that inequality (2.7) requires that $|D(0)| < M$. For the case $D(0) = 0$, inequality (2.7) can be written as

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec Mz \qquad \Rightarrow \qquad p(z) \prec Nz.$$

In [35, Lemma 2.2d, p. 24], if p is not subordinate to q, it is evident that there exist a point $z_0 \in \mathbb{U}$, for which $p(\{z : |z| < |z_0|\}) \subset q(\mathbb{U})$. By using the similar approach, this theorem will be proved by making use of Lemma 2.1.

Suppose that $|p(z)| \geq N$ and let

$$W(z) = B(z)zp'(z) + C(z)p(z) + D(z).$$

Applying Lemma 2.1 with $q(z) = Nz$, there exists a point $z_0 \in \mathbb{U}$, $\zeta_0 \in \partial\mathbb{U}$ and an $m \geq n + (N - \beta|z_0|^n)/(N + \beta|z_0|^n)$ such that $p(z_0) = N\zeta_0$ and $z_0 p'(z_0) = mN\zeta_0$. Thus

$$\begin{aligned}
|W(z_0)| &= |B(z_0)z_0p'(z_0) + C(z_0)p(z_0) + D(z_0)| \\
&= |B(z_0)mN\zeta_0 \ + \ C(z_0)N\zeta_0 + D(z_0)| \\
&= |N(mB(z_0) \ + \ C(z_0)) \ + \ \overline{\zeta_0}D(z_0)| \\
&\geq N|mB(z_0) \ + \ C(z_0)| \ - \ |D(z_0)|.
\end{aligned}$$

Consider the function $g : [0,1] \to \mathbb{R}$ given by

$$g(r) = \frac{N - \beta r^n}{N + \beta r^n}, \qquad N > 0,\ \beta > 0,\ 0 < r^n < 1,$$

then g is a continuous function of r and

$$g'(r) = \frac{-2nN\beta r^{n-1}}{(N + \beta r^n)^2} \leq 0.$$

Therefore, g is a decreasing function of r and so

$$1 \ \geq \ \frac{N - \beta r^n}{N + \beta r^n} \ \geq \ \frac{N - \beta}{N + \beta}.$$

since $g(0) \geq g(r) \geq g(1)$. In this case,

$$m \ \geq \ n + \frac{N - \beta|z_0|^n}{N + \beta|z_0|^n} \ \geq \ n + \frac{N - \beta}{N + \beta}.$$

Now condition (i) gives

$$0 \leq \left(n + \frac{N - \beta}{N + \beta}\right) + \mathrm{Re}\,\left(\frac{C(z)}{B(z)}\right) \leq \left|\left(n + \frac{N - \beta}{N + \beta}\right) + \frac{C(z)}{B(z)}\right| \leq \left|m + \frac{C(z)}{B(z)}\right|,$$

where the latter inequality follows from $m \geq n + (N-\beta)/(N+\beta)$. Therefore,

$$|mB(z) + C(z)| \geq \left| \left(n + \frac{N-\beta}{N+\beta} \right) B(z) + C(z) \right|,$$

which then by condition (ii) yields

$$|mB(z) + C(z)| \geq \frac{1}{N}(M + |D(z)|).$$

Thus

$$|W(z_0)| \geq N|mB(z_0) + C(z_0)| - |D(z_0)| \geq M,$$

which contradicts inequality (2.7), and this yields the desired result. $\square$

For $\beta = |q'(0)| = N$, Theorem 2.2 reduces to Theorem 4.1b in [35].

Corrollary 2.3. [35, Theorem 4.1b, p. 190] *Let $M > 0$, $N > 0$, and let n be a positive integer. Suppose that the functions $B, C, D : \mathbb{U} \to \mathbb{C}$ satisfy $B(z) \neq 0$,*

(i) $\operatorname{Re}\left(\frac{C(z)}{B(z)}\right) \geq -n$, *and*

(ii) $|nB(z) + C(z)| \geq \frac{1}{N}(M + |D(z)|)$.

If $p \in \mathcal{H}[0, n]$ satisfies

$$\left| B(z)zp'(z) + C(z)p(z) + D(z) \right| < M,$$

then $|p(z)| < N$.

In the case $D(z) \equiv 0$, it follows from condition (ii) of Theorem 2.2 that

$$\begin{aligned}
\frac{\left| \left(n + \frac{N-\beta}{N+\beta} \right) B(z) + C(z) \right|}{|B(z)|} &= \left| \left(n + \frac{N-\beta}{N+\beta} \right) + \frac{C(z)}{B(z)} \right| \\
&\geq \left| \frac{C(z)}{B(z)} \right| \geq \left| \operatorname{Im}\left(\frac{C(z)}{B(z)} \right) \right| \geq \frac{M}{N|B(z)|}.
\end{aligned}$$

This shows that the latter inequality does not depend on n. Therefore, Theorem 2.2 is improved by replacing conditions (i) and (ii) with the following conditions:

Theorem 2.3. *Let $M > 0,\ N > 0$, and suppose that $B, C : \mathbb{U} \to \mathbb{C}$ satisfy $B(z) \neq 0$, and*

$$\left| \operatorname{Im} \left(\frac{C(z)}{B(z)} \right) \right| \geq \frac{M}{N|B(z)|}. \tag{2.8}$$

If $p \in \mathcal{H}_\beta[0,1],\ 0 < \beta \leq N$, satisfies

$$\left| B(z)zp'(z) + C(z)p(z) \right| < M,$$

then $|p(z)| < N$.

Proof. Following the proof of Theorem 2.2, suppose that $|p(z)| \geq N$ and let

$$W(z) = B(z)zp'(z) + C(z)p(z).$$

According to Lemma 2.1, there exists a point $z_0 \in \mathbb{U},\ \zeta_0 \in \partial\mathbb{U}$ and an $m \geq n + (N - \beta)/(N+\beta)$ so that

$$|W(z_0)| = |mN\zeta_0 B(z_0) + N\zeta_0 C(z_0)| = N|mB(z_0) + C(z_0)|.$$

Consequently,

$$|W(z_0)|^2 - M^2 = \left[N^2|B(z_0)|^2 m^2 + 2N^2 \operatorname{Re}\,(\overline{B(z_0)}C(z_0))m + N^2|C(z_0)|^2 \right] - M^2.$$

Condition (2.8) implies that the above quadratic expression in m has a non-positive discriminant, i.e.,

$$\begin{aligned} b^2 - 4ac &= \left(2N^2 \operatorname{Re}\,(\overline{B(z_0)}C(z_0)) \right)^2 - 4(N^2|B(z_0)|^2)(N^2|C(z_0)|^2 - M^2) \\ &= 4N^4 \left(\operatorname{Re}\,(\overline{B(z_0)}C(z_0)) \right)^2 - 4N^4|B(z_0)|^2|C(z_0)|^2 + 4M^2N^2|B(z_0)|^2 \\ &= 4N^4|B(z_0)|^4 \left[\frac{\left(\operatorname{Re}\,(\overline{B(z_0)}C(z_0)) \right)^2}{|B(z_0)|^4} - \frac{|C(z_0)|^2}{|B(z_0)|^2} + \frac{M^2}{N^2|B(z_0)|^2} \right] \\ &= 4N^4|B(z_0)|^4 \left[\left| \operatorname{Re} \left(\frac{C(z_0)}{B(z_0)} \right) \right|^2 - \frac{|C(z_0)|^2}{|B(z_0)|^2} + \frac{M^2}{N^2|B(z_0)|^2} \right] \end{aligned}$$

By squaring the condition (2.8) and applying it to the latter inequality yields

$$b^2 - 4ac \ \leq \ 4N^4|B(z_0)|^4 \left[\left|\text{Re}\left(\frac{C(z_0)}{B(z_0)}\right)\right|^2 - \frac{|C(z_0)|^2}{|B(z_0)|^2} + \left|\text{Im}\left(\frac{C(z_0)}{B(z_0)}\right)\right|^2 \right].$$

When $(|C(z_0)|/|B(z_0)|)^2 = |\text{Re}\,(C(z_0)/B(z_0))|^2 + |\text{Im}\,(C(z_0)/B(z_0))|^2$, the right side of the above inequality becomes 0 and this shows that the discriminant is non-positive. Since the coefficient of m^2 is positive, it follows that $|W(z_0)|^2 - M^2 \geq 0$ or $|W(z_0)| \geq M$, which contradicts the hypothesis. Therefore, $|p(z)| < N$. □

For $\beta = |q'(0)| = N$, Theorem 2.3 easily reduces to Theorem 4.1c in [35].

Corrollary 2.4. [35, Theorem 4.1c, p. 192] *Let $M > 0$, $N > 0$, and suppose that $B, C : \mathbb{U} \to \mathbb{C}$ satisfy $B(z) \neq 0$, and*

$$\left|\text{Im}\left(\frac{C(z)}{B(z)}\right)\right| \geq \frac{M}{N|B(z)|}.$$

If $p \in \mathcal{H}[0,1]$ and

$$|B(z)zp'(z) + C(z)p(z)| < M,$$

then $|p(z)| < N$.

By using inequality (2.8), a bound N for $|p(z)|$ can be determined. This is established by solving (2.8) for N, that is,

$$N \geq \frac{M}{|B(z)| \cdot \left|\text{Im}\left(\frac{C(z)}{B(z)}\right)\right|},$$

and taking the supremum of the expansion of the right over $\mathbb{U}$. When the supremum is finite yields the following corollary.

Corollary 2.5. *Let $M > 0$ and $B, C : \mathbb{U} \to \mathbb{C}$ with $B(z) \neq 0$. If $p \in \mathcal{H}_\beta[0,1]$, $0 < \beta \leq$*

N, and

$$N=\sup_{|z|<1}\left\{\frac{M}{|B(z)|\cdot\left|\operatorname{Im}\left(\frac{C(z)}{B(z)}\right)\right|}\right\}<+\infty, \tag{2.9}$$

then

$$|B(z)zp'(z)+C(z)p(z)|<M \qquad \Rightarrow \qquad |p(z)|<N.$$

The next example illustrates the similar result of Corollary 2.5.

Example 2.2. *Let* $M=1,\ B(z)=1$ *and* $C(z)=4i+2z$*, then*

$$\frac{1}{|\operatorname{Im}(4i+2z)|}\leq\frac{1}{2},$$

since

$$|\operatorname{Im}(4i+2z)|=|\operatorname{Im}(4i)+\operatorname{Im}(2z)|\geq|\operatorname{Im}(4i)|-|\operatorname{Im}(2z)|\geq 4-2=2.$$

In this case, condition (2.9) in Corollary 2.5 yields

$$N=\sup_{|z|<1}\left\{\frac{1}{|\operatorname{Im}(4i+2z)|}\right\}=\frac{1}{2}.$$

The result that follows from Corollary 2.5 is as follows:

If $p\in\mathcal{H}_\beta[0,1],\ 0<\beta\leq 1/2$*, then*

$$|zp'(z)+(4i+2z)p(z)|<1 \qquad \Rightarrow \qquad |p(z)|<\frac{1}{2}.$$

Let Ω in (2.4) still be the disk of radius $M>0$ centered at the origin but now consider the case when $A(z)\neq 0$.

Theorem 2.4. *Let* $M>0,\ N>0,\ 0<\beta\leq N$*, and n be a positive integer. Suppose* $A,B,C,D:\mathbb{U}\to\mathbb{C}$ *satisfy* $A(z)\neq 0$*,*

(i) $\operatorname{Re}\left(\frac{B(z)}{A(z)}\right)\geq-\left(n+\frac{N-\beta}{N+\beta}\right)$*, and*

(ii) Re $\left(\overline{A(z)}\,(B(z)+C(z))\right) \geq \frac{|A(z)|}{N}(M+|D(z)|)$.

If $p \in \mathcal{H}_{\beta}[0,n]$ *and*

$$\left|A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\right| < M, \tag{2.10}$$

then $|p(z)| < N$.

Proof. Note that inequality (2.10) required that $|D(0)| < M$. Let

$$\psi(r,s,t;z) = A(z)t+B(z)s+C(z)r+D(z),$$

$\Omega = \{w : |w| < M\}$, and $q(z) = Nz$. The object is to show that $\psi \in \Psi_{n,\beta}(\Omega,N,0)$. The function ψ is continuous in a domain $\mathrm{D} = \mathbb{C}^3 \times \mathbb{U}$, $(0,0,0;0) \in \mathrm{D}$ and $|\psi(0,0,0;0)| = |D(0)| < M$ so that $\psi(0,0,0;0) \in \Omega$. The β-admissibility condition (2.3) is satisfied if

$$|\psi(z)| := |\psi(Ne^{i\theta},Ke^{i\theta},L)| \geq M$$

whenever $K \geq mN$, and Re $\left(Le^{-i\theta}\right) \geq [n-2\beta/(N+\beta)]K$, with $m \geq n+(N-\beta)/(N+\beta)$, $n \geq 1$, and $\theta \in \mathbb{R}$.

Now,

$$\begin{aligned}
\frac{|\psi(z)|}{|A(z)|} &= \frac{|A(z)L+B(z)Ke^{i\theta}+C(z)Ne^{i\theta}+D(z)|}{|A(z)|} \\
&\geq \left|Le^{-i\theta}+\frac{B(z)}{A(z)}K+\frac{C(z)}{A(z)}N\right| - \left|\frac{D(z)}{A(z)}e^{-i\theta}\right| \\
&\geq \text{Re}\left(Le^{-i\theta}\right)+\text{Re}\left(\frac{B(z)}{A(z)}\right)mN+\text{Re}\left(\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right| \\
&\geq \left(n-\frac{2\beta}{N+\beta}\right)mN+\text{Re}\left(\frac{B(z)}{A(z)}\right)mN-\text{Re}\left(\frac{B(z)}{A(z)}\right)N \\
&\qquad +\text{Re}\left(\frac{B(z)}{A(z)}\right)N+\text{Re}\left(\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right| \\
&= \left(n-\frac{2\beta}{N+\beta}\right)mN+\text{Re}\left(\frac{B(z)}{A(z)}\right)(m-1)N \\
&\qquad +\text{Re}\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right|.
\end{aligned}$$

By using condition (i) in the last inequality yields

$$\begin{aligned}
\frac{|\psi(z)|}{|A(z)|} &\geq \left(n-\frac{2\beta}{N+\beta}\right)mN-\left(n+\frac{N-\beta}{N+\beta}\right)(m-1)N \\
&\qquad +\mathrm{Re}\,\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right| \\
&= mnN-\left(\frac{2\beta}{N+\beta}\right)mN-mnN+nN-\left(\frac{N-\beta}{N+\beta}\right)mN \\
&\qquad +\left(\frac{N-\beta}{N+\beta}\right)N+\mathrm{Re}\,\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right| \\
&= -mN+\left(n+\frac{N-\beta}{N+\beta}\right)N+\mathrm{Re}\,\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right| \\
&\geq -\left(n+\frac{N-\beta}{N+\beta}\right)N+\left(n+\frac{N-\beta}{N+\beta}\right)N+\mathrm{Re}\,\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right|,
\end{aligned}$$

where the latter inequality follows from $m \geq n+(N-\beta)/(N+\beta)$. This leads to

$$\frac{|\psi(z)|}{|A(z)|} \geq \mathrm{Re}\,\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-\left|\frac{D(z)}{A(z)}\right|.$$

Thus

$$|A(z)|^2\frac{|\psi(z)|}{|A(z)|} \geq |A(z)|^2\mathrm{Re}\,\left(\frac{B(z)}{A(z)}+\frac{C(z)}{A(z)}\right)N-|A(z)|^2\left|\frac{D(z)}{A(z)}\right|.$$

It follows from condition (ii) that

$$|A(z)||\psi(z)| \geq \frac{|A(z)|}{N}(M+|D(z)|)N-|A(z)||D(z)| = |A(z)|M,$$

which shows that $|\psi(z)| \geq M$, and whence, $\psi \in \Psi_{n,\beta}(\Omega,N,0)$. Therefore, Lemma 2.3(i) deduces the required result. □

For $\beta = |q'(0)| = N$, Theorem 2.4 reduces to Theorem 4.1d in [35].

Corrollary 2.6. [35, Theorem 4.1d, p. 193] *Let $M>0$, $N>0$ and let n be a positive* *integer. Suppose that the functions* $A, B, C, D : \mathbb{U} \to \mathbb{C}$ *satisfy* $A(z) \neq 0$,

(i) $\operatorname{Re}\left(\frac{B(z)}{A(z)}\right) \geq -n$, *and*

(ii) $\operatorname{Re}\left(\overline{A(z)}\,(B(z)+C(z))\right) \geq \frac{|A(z)|}{N}(M+|D(z)|)$.

If $p \in \mathcal{H}[0,n]$, *and*

$$\left|A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\right| < M,$$

then $|p(z)| < N$.

In this case, a bound N for $|p(z)|$ can also be determined so that the inequality (2.10) implies $|p(z)| < N$. By using a similar technique as in the previous case, solve condition (ii) in Theorem 2.4 for N, that is,

$$N \geq \frac{|A(z)|(M+|D(z)|)}{\operatorname{Re}\left(\overline{A(z)}\,(B(z)+C(z))\right)},$$

and take the supremum of the expansion of the right over $\mathbb{U}$. When the supremum is finite it leads to the following corollary.

Corrollary 2.7. *Let* $M > 0$, $N > 0$ *and* n *be a positive integer. If* $p \in \mathcal{H}_\beta[0,n]$, $0 < \beta \leq N$ *and that* $A, B, C, D : \mathbb{U} \to \mathbb{C}$, *with* $A(z) \neq 0$ *satisfy*

$$\operatorname{Re}\left(\frac{B(z)}{A(z)}\right) \geq -\left(n+\frac{N-\beta}{N+\beta}\right), \qquad \operatorname{Re}\left(\overline{A(z)}(B(z)+C(z))\right) \geq 0,$$

and

$$N = \sup_{|z|<1}\left\{\frac{|A(z)|(M+|D(z)|)}{\operatorname{Re}\left(\overline{A(z)}\,(B(z)+C(z))\right)}\right\} < +\infty,$$

then

$$\left|A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\right| < M \quad \Rightarrow \quad |p(z)| < N.$$

2.4 Differential Subordination by Convex Functions

The inclusion (2.4) can also be written as

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z).$$

Thus far, the function h considered is either the right-half plane, $h(z)=(1+z)/(1-z)$ in Section 2.2 or a disk of radius $M>0$ defined by $h(z)=Mz$ in Section 2.3. Now, let h be any convex univalent function of order α.

Theorem 2.5. *Let n be a positive integer. For $0\leq\alpha<1$, let h be a convex univalent function of order α in $\mathbb{U}$ with $h(0)=0$. Further, let $0<\beta\leq|h'(0)|$ and $k>2^{2(1-\alpha)}/|h'(0)|$. Suppose that $A\geq 0$ and that $B,C,$ and D are analytic functions in $\mathbb{U}$ satisfying*

$$\begin{aligned}\text{Re } B(z)\geq\left[1-\alpha\left(n+\frac{|h'(0)|-\beta}{|h'(0)|+\beta}\right)\right]A+\left(\frac{|h'(0)|+\beta}{(n+1)|h'(0)|+(n-1)\beta}\right)\\ \times\left[\frac{1}{2\tau(\alpha)}|C(z)-1|-\frac{1}{2\tau(\alpha)}\text{Re }\big(C(z)-1\big)+k|D(z)|\right],\end{aligned}\tag{2.11}$$

where

$$\tau(\alpha):=\begin{cases}\frac{2\alpha-1}{2-2^{2(1-\alpha)}}, & if \quad \alpha\neq\frac{1}{2},\\[2ex] \frac{1}{2\ln 2}, & if \quad \alpha=\frac{1}{2}.\end{cases}\tag{2.12}$$

If $p\in\mathcal{H}_\beta[0,n]$ satisfies the differential subordination

$$Az^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z),\tag{2.13}$$

then $p(z)\prec h(z)$.

Proof. Note that subordination (2.13) implies that $D(0)=0$. Let $t(\rho_0)=(1+\rho_0)^{2(1-\alpha)}/\rho_0$. Since the function t is decreasing on $(0,1)$, it follows that there is an $0<\rho_0<1$ satisfying

$$\frac{(1+\rho_0)^{2(1-\alpha)}}{\rho_0}=k|h'(0)|,$$

and

$$2^{2(1-\alpha)} < \frac{(1+\rho)^{2(1-\alpha)}}{\rho} < k|h'(0)|$$

for $\rho_0 < \rho < 1$.

Since h is convex of order α in $\mathbb{U}$, it follows that $h_\rho(z) := h(\rho z)$ is convex of order α in $\overline{\mathbb{U}}$ for $\rho_0 < \rho < 1$. Let $p_\rho(z) := p(\rho z)$ for $\rho_0 < \rho < 1$. Evidently,

$$p'_\rho(z) = \rho p'(\rho z), \qquad \text{and} \qquad p''_\rho(z) = \rho^2 p''(\rho z). \tag{2.14}$$

From the definition of subordination, the subordination (2.13) is equivalent to

$$Az^2 p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) = h(\phi(z))$$

where ϕ is an analytic self-map of $\mathbb{U}$ with $\phi(0) = 0$. A change of variable z with ρz reduces the above equality to

$$Az^2\rho^2 p''(\rho z) + B(\rho z)z\rho p'(\rho z) + C(\rho z)p(\rho z) + D(\rho z) = h\big(\phi(\rho z)\big).$$

In view of the fact that $h_\rho(z) = h(\rho z)$ and (2.14), it follows that

$$Az^2 p''_\rho(z) + B(\rho z)zp'_\rho(z) + C(\rho z)p_\rho(z) + D(\rho z) = h_\rho\big(\phi(z)\big),$$

which leads to

$$J_\rho(z) := Az^2 p''_\rho(z) + B(\rho z)zp'_\rho(z) + C(\rho z)p_\rho(z) + D(\rho z) \prec h_\rho(z), \tag{2.15}$$

for $\rho_0 < \rho < 1$ and $z \in \mathbb{U}$.

The purpose now is to show that $p_\rho(z) \prec h_\rho(z)$ for all ρ satisfying $\rho_0 < \rho < 1$. Assume that p_ρ is not subordinate to h_ρ for some $\rho_0 < \rho < 1$. According by Lemma 2.1, there exist points $z_0 \in \mathbb{U}$, $\zeta_0 \in \partial\mathbb{U}$, and an $m \geq n + (|h'(0)| - \beta|z_0|^n)/(|h'(0)| + \beta|z_0|^n)$ such that

$$p_\rho(z_0) = h_\rho(\zeta_0), \tag{2.16}$$

$$z_0 p'_\rho(z_0) = m\zeta_0 h'_\rho(\zeta_0), \tag{2.17}$$

and

$$\operatorname{Re}\left(1+\frac{z_0 p''_\rho(z_0)}{p'_\rho(z_0)}\right) \geq m\operatorname{Re}\left(1+\frac{\zeta_0 h''_\rho(\zeta_0)}{h'_\rho(\zeta_0)}\right).$$

In view of the fact that h_ρ is convex of order α, it is clearly that

$$\operatorname{Re}\left(1+\frac{z_0 p''_\rho(z_0)}{p'_\rho(z_0)}\right) \geq m\alpha.$$

Using p'_ρ from (2.17) yields

$$\operatorname{Re}\left(1+\frac{{z_0}^2 p''_\rho(z_0)}{m\zeta_0 h'_\rho(\zeta_0)}\right) \geq m\alpha,$$

and a straightforward computation shows that

$$\operatorname{Re}\left(\frac{{z_0}^2 p''_\rho(z_0)}{\zeta_0 h'_\rho(\zeta_0)}\right) \geq m(m\alpha-1). \tag{2.18}$$

Now, consider the function $g : [0,1] \to \mathbb{R}$ given by

$$g(r) = \frac{|h'(0)|-\beta r^n}{|h'(0)|+\beta r^n}, \qquad |h'(0)| \geq \beta,\ \beta > 0,\ 0 < r^n < 1,$$

then g is a continuous function of r and

$$g'(r) = \frac{-2|h'(0)|n\beta r^{n-1}}{(|h'(0)|+\beta r^n)^2} \leq 0.$$

Therefore, g is a decreasing function of r and

$$1 \ \geq \ \frac{|h'(0)|-\beta r^n}{|h'(0)|+\beta r^n} \ \geq \ \frac{|h'(0)|-\beta}{|h'(0)|+\beta}.$$

since $g(0) \geq g(r) \geq g(1)$. In this case,

$$m \;\geq\; n + \frac{|h'(0)| - \beta |z_0|^n}{|h'(0)| + \beta |z_0|^n} \;\geq\; n + \frac{|h'(0)| - \beta}{|h'(0)| + \beta}.$$

Let

$$\lambda = \frac{J_\rho(z_0) - h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)},$$

or equivalently,

$$J_\rho(z_0) = h_\rho(\zeta_0) + \lambda \zeta_0 h'_\rho(\zeta_0), \tag{2.19}$$

then the definition of J_ρ in (2.15) readily gives

$$\lambda = \frac{\Big(A{z_0}^2 p''_\rho(z_0) + B(\rho z_0) z_0 p'_\rho(z_0) + C(\rho z_0) p_\rho(z_0) + D(\rho z_0)\Big) - h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)}.$$

It follows from (2.16) that

$$\begin{aligned}
\lambda &= \frac{\Big(A{z_0}^2 p''_\rho(z_0) + B(\rho z_0) z_0 p'_\rho(z_0) + C(\rho z_0) h_\rho(\zeta_0) + D(\rho z_0)\Big) - h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)} \\
&= \frac{A{z_0}^2 p''_\rho(z_0)}{\zeta_0 h'_\rho(\zeta_0)} + \frac{B(\rho z_0) z_0 p'_\rho(z_0)}{\zeta_0 h'_\rho(\zeta_0)} + \frac{(C(\rho z_0) - 1)\, h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)} + \frac{D(\rho z_0)}{\zeta_0 h'_\rho(\zeta_0)}\ .
\end{aligned}$$

Thus

$$\begin{aligned}
\text{Re }\lambda \geq A\,\text{Re}\left(\frac{{z_0}^2 p''_\rho(z_0)}{\zeta_0 h'_\rho(\zeta_0)}\right) &+ \text{Re}\left(B(\rho z_0)\frac{z_0 p'_\rho(z_0)}{\zeta_0 h'_\rho(\zeta_0)}\right) \\
&+ \text{Re}\left(\frac{(C(\rho z_0) - 1) h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)}\right) - \left|\frac{D(\rho z_0)}{\zeta_0 h'_\rho(\zeta_0)}\right|.
\end{aligned} \tag{2.20}$$

To get the desired contradiction, it is enough to show that Re $\lambda > 0$. According to Theorem 1.15 (see Section 1.2), when h_ρ is convex of order α, then

$$\text{Re}\left(\frac{\zeta_0 h'_\rho(\zeta_0)}{h_\rho(\zeta_0)}\right) \geq \tau(\alpha), \qquad (\zeta_0 \in \overline{\mathbb{U}}),$$

where $\tau(\alpha)$ is given by equation (2.12). Equivalently,

$$\left|\frac{h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)} - \frac{1}{2\tau(\alpha)}\right| \leq \frac{1}{2\tau(\alpha)} \qquad \text{or} \qquad \left|\frac{2\tau(\alpha)h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)} - 1\right| \leq 1.$$

If $R, S \in \mathbb{C}$ and $|S-1| \leq 1$, then

$$\text{Re } RS = \text{Re } R + \text{Re } R(S-1) \geq \text{Re } R - |R|.$$

Choosing $R = C(\rho z_0) - 1$ and $S = \left(2\tau(\alpha)h_\rho(\zeta_0)\right)/\zeta_0 h'_\rho(\zeta_0)$ in the above inequality yields

$$\text{Re }\left((C(\rho z_0)-1)\frac{2\tau(\alpha)h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)}\right) \geq \text{Re }(C(\rho z_0)-1) - |C(\rho z_0)-1|,$$

that is,

$$\text{Re }\left(\frac{(C(\rho z_0)-1)h_\rho(\zeta_0)}{\zeta_0 h'_\rho(\zeta_0)}\right) \geq \frac{1}{2\tau(\alpha)}\Big[\text{Re }(C(\rho z_0)-1) - |C(\rho z_0)-1|\Big]. \tag{2.21}$$

Since h is convex of order α with $h(0) = 0$, then a well-known distortion theorem for convex functions of order α (see Theorem 1.16) applied to $h'(z)/h'(0)$ yields

$$|h'(z)| \geq \frac{|h'(0)|}{(1+\rho)^{2(1-\alpha)}}, \qquad (|z| = \rho < 1).$$

Replacing z by $\rho\zeta_0$ in the above inequality gives

$$|h'(\rho\zeta_0)| \geq \frac{|h'(0)|}{(1+\rho)^{2(1-\alpha)}},$$

which leads to

$$\left|\frac{h'_\rho(\zeta_0)}{\rho}\right| \geq \frac{|h'(0)|}{(1+\rho)^{2(1-\alpha)}}$$

because $\rho h'(\rho\zeta_0) = h'_\rho(\zeta_0)$. Since $|\zeta_0| = 1$, it follows that

$$|\zeta_0 h'_\rho(\zeta_0)| \geq \frac{\rho|h'(0)|}{(1+\rho)^{2(1-\alpha)}}. \tag{2.22}$$

In view of (2.17),(2.18),(2.21) and (2.22), it follows from (2.20) that

$$\begin{aligned}\text{Re }\lambda \geq\, & m(m\alpha-1)A + m\text{Re }B(\rho z_0) + \frac{1}{2\tau(\alpha)}\Big[\text{Re }(C(\rho z_0)-1) - |C(\rho z_0)-1|\Big] \\ & - \frac{(1+\rho)^{2(1-\alpha)}}{\rho|h'(0)|}|D(\rho z_0)|\end{aligned}$$

Substituting $m \geq n + (|h'(0)|-\beta)/(|h'(0)|+\beta)$ and $k \geq (1+\rho)^{2(1-\alpha)}/(\rho|h'(0)|)$ in the above inequality yields

$$\begin{aligned}\text{Re }\lambda \geq\, & \left(n + \frac{|h'(0)|-\beta}{|h'(0)|+\beta}\right)\left[\left(n + \frac{|h'(0)|-\beta}{|h'(0)|+\beta}\right)\alpha - 1\right]A \\ & + \left(n + \frac{|h'(0)|-\beta}{|h'(0)|+\beta}\right)\text{Re }B(\rho z_0) \\ & + \frac{1}{2\tau(\alpha)}\Big[\text{Re }(C(\rho z_0)-1) - |C(\rho z_0)-1|\Big] - k|D(\rho z_0)| \geq 0\end{aligned}$$

provided condition (2.11) holds. Note that Re $\lambda \geq 0$ is equivalent to $|\arg\lambda| < \pi/2$. This together with the fact that $\zeta_0 h'_\rho(\zeta_0)$ is the outward normal to the boundary of the convex domain $h_\rho(\mathbb{U})$ at $h_\rho(\zeta_0)$, it follows from (2.19) that $J_\rho(z_0) \notin h_\rho(\mathbb{U})$ for some ρ satisfying $\rho_0 < \rho < 1$. This contradicts the assumption (2.15) and hence, $p_\rho(z) \prec h_\rho(z)$ for all ρ satisfying $\rho_0 < \rho < 1$. Therefore, by letting $\rho \to 1^-$ yields the desired result, which completes the proof of this theorem. □

Note that various earlier published results are special cases of Theorem 2.5 for suitable choices of the parameters α, β and n, and the functions A, B, C and D. First, the case $\beta = |h'(0)|$ corresponds to the following result of Ravichandran [51].

Corrollary 2.8. [51, Theorem 2.1, p. 4] *Let h be a convex univalent function of order*

α, $0 \le \alpha < 1$, in $\mathbb{U}$ with $h(0) = 0$ and let $A \ge 0$. Suppose that $k > 2^{2(1-\alpha)}/|h'(0)|$, and that B, C, and D are analytic function in $\mathbb{U}$ and satisfy

$$n\mathrm{Re}\ B(z) \ge n(1-\alpha n)A + \frac{1}{2\tau(\alpha)}\Big(|C(z)-1| - \mathrm{Re}\ \big(C(z)-1\big)\Big) + k|D(z)|,$$

where $\tau(\alpha)$ is as given by equation (2.12). If $p \in \mathcal{H}[0,n]$ satisfies

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Further, the case where $\alpha = 0$ and $n = 1$ leads to the following result of Miller and Mocanu [35].

Corrollary 2.9. [35, Theorem 4.1e, p. 195] *Let h be a convex in $\mathbb{U}$ with $h(0) = 0$ and let $A \ge 0$. Suppose that $k > 4/|h'(0)|$, and that B, C, and D are analytic in $\mathbb{U}$ and satisfy*

$$\mathrm{Re}\ B(z) \ge A + |C(z)-1| - \mathrm{Re}\ \big(C(z)-1\big) + k|D(z)|,$$

If $p \in \mathcal{H}[0,1]$ satisfies

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Next, in the case $A = 0,\ D(z) \equiv 0$, and $\beta = |h'(0)|$, Theorem 2.5 yields a result of Ravichandran [51].

Corrollary 2.10. [51, Corollary 2.4, p. 6] *Let h be a convex univalent function of order α, $0 \le \alpha < 1$, in $\mathbb{U}$, with $h(0) = 0$. Let B and C are analytic on $\mathbb{U}$ satisfying*

$$\mathrm{Re}\ B(z) \ge \frac{1}{2n\tau(\alpha)}\Big(|C(z)-1| - \mathrm{Re}\ \big(C(z)-1\big)\Big)$$

where $\tau(\alpha)$ is as given by equation (2.12). If $p \in \mathcal{H}[0,n]$ satisfies

$$B(z)zp'(z) + C(z)p(z) \prec h(z),$$

then $p(z) \prec h(z)$.

Finally, for the particular case where $A = 0,\ B(z) = 1,\ D(z) \equiv 0,\ \alpha = 0,\ n = 1,$ and $\beta = |h'(0)|$, Theorem 2.5 reduces to the following result of Ravichandran [51].

Corollary 2.11. [51, Corollary 2.5, p. 6] *Let h be a convex univalent function in $\mathbb{U}$ with $h(0) = 0$. Let C be analytic functions on $\mathbb{U}$ satisfying*

$$\mathrm{Re}\ C(z) > |C(z) - 1|.$$

If $p \in \mathcal{H}[0,1]$ satisfies

$$zp'(z) + C(z)p(z) \prec h(z),$$

then $p(z) \prec h(z)$.

For the case $C(z) \equiv 1$, the condition $h(0) = p(0) = 0$ can be replaced by $h(0) = p(0) = a$ and the proof of Theorem 2.5 still holds. In this case, the following result is stated without proof.

Theorem 2.6. *Let n be a positive integer. For $0 \leq \alpha < 1$, let h be convex univalent function of order α in $\mathbb{U}$ with $h(0) = a$. Further, let $0 < \beta \leq |h'(0)|$ and $k > 2^{2(1-\alpha)}/|h'(0)|$. Suppose that $A \geq 0$ and that B and D are analytic in $\mathbb{U}$ with $D(0) = 0$ satisfying*

$$\mathrm{Re}\ B(z) \geq \left[1 - \left(n + \frac{|h'(0)| - \beta}{|h'(0)| + \beta}\right)\alpha\right]A + \left(\frac{|h'(0)| + \beta}{(n+1)|h'(0)| + (n-1)\beta}\right)k|D(z)|.$$

If $p \in \mathcal{H}_\beta[a,n]$, satisfies the differential subordination

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z),$$

then $p(z) \prec h(z)$.

For the case $\beta = |h'(0)|$, Theorem 2.6 reduces to the following result of Ravichandran [51].

Corollary 2.12. [51, Corollary 2.3, p. 6] *Let h be a convex univalent function of order α, $0 \leq \alpha < 1$, in $\mathbb{U}$ and let $A \geq 0$. Suppose that $k > 2^{2(1-\alpha)}/|h'(0)|$ and that B and D are analytic in $\mathbb{U}$ with $D(0) = 0$ and*

$$n\text{Re } B(z) \geq n(1-\alpha n)A + k|D(z)|$$

for all $z \in \mathbb{U}$. If $p \in \mathcal{H}[a,n]$, $h(0) = p(0)$ satisfies the differential subordination

$$Az^2 p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z),$$

then $p(z) \prec h(z)$.

For the particular case where $\alpha = 0$, $n = 1$ and $\beta = |h'(0)|$, Theorem 2.6 coincides to the following result of Miller and Mocanu [35].

Corollary 2.13. [35, Theorem 4.1f, p. 198] *Let h be a convex in $\mathbb{U}$ and $A \geq 0$. Suppose that B and D are analytic in $\mathbb{U}$ with $D(0) = 0$ satisfying*

$$\text{Re } B(z) \geq A + 4\frac{|D(z)|}{|h'(0)|}.$$

If $p \in \mathcal{H}[h(0), 1]$ satisfies

$$Az^2 p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z),$$

then $p(z) \prec h(z)$.

CHAPTER 3

LINEAR INTEGRAL OPERATORS ON ANALYTIC FUNCTIONS WITH FIXED INITIAL COEFFICIENT

3.1 Introduction

During the last several years many authors have employed various methods to study different types of integral operators $\mathbf{I}$ mapping subsets of $\mathcal{A}$ into $\mathcal{A}$. Some interesting development of integral operator involving subordination were considered by Ali et al. [4, 5, 6] and others [14, 16, 30, 38, 58]. Motivated by these results, this chapter studies the inclusion properties for linear integral operators on certain subclasses of analytic functions with fixed initial coefficient by using the methodology of differential subordination.

In the present investigation, several new results of the linear integral operators that preserve certain subclasses of analytic functions with fixed initial coefficient are obtained. These subclasses include the function with positive real part, bounded functions, and convex functions. The linear integral operator is shown to map certain subclasses of analytic functions with fixed initial coefficient into itself. To ensure these inclusion properties hold, conditions on the linear integral operators are determined by satisfying certain equivalent conditions from the previous chapter. The methodology used rests on differential subordination by making modifications and improvements to the works developed in [35].

Denote by $\mathcal{D}_n$ the class consisting of nonzero analytic functions given by

$$\mathcal{D}_n := \{\varphi \in \mathcal{H}[1,n] : \varphi(z) \neq 0 \quad \text{for} \quad z \in \mathbb{U}\}$$

with $\mathcal{D}_1 := \mathcal{D}$. This class of nonzero analytic functions will be used frequently in the remainder of this chapter.

3.2 Integral Operators Preserving Functions with Positive Real Part

Denote by $\mathcal{P}_n$ the class of analytic functions with positive real part given by

$$\mathcal{P}_n := \{f \in \mathcal{H}[1,n] : \text{Re } f(z) > 0 \quad \text{for} \quad z \in \mathbb{U}\}.$$

For $\beta > 0$, consider the subclass $\mathcal{P}_{n,\beta}$ of $\mathcal{P}_n$, consisting of all analytic functions in $\mathbb{U}$ having positive real part:

$$\mathcal{P}_{n,\beta} := \{f \in \mathcal{H}_\beta[1,n] : f(z) = 1 + \beta z^n + f_{n+1}z^{n+1} + \cdots, \text{ Re } f(z) > 0\}.$$

Also, consider the subclass $\mathcal{D}_{n,-\beta}$ of $\mathcal{D}_n$, consisting of all nonzero analytic functions in $\mathbb{U}$ given by

$$\mathcal{D}_{n,-\beta} := \{\varphi \in \mathcal{H}_{-\beta}[1,n] : \varphi(z) = 1 - \beta z^n + \varphi_{n+1}z^{n+1} + \cdots, \ \varphi(z) \neq 0\}.$$

A result from Chapter 2 which is Corollary 2.2 is required.

Lemma 3.1. (Corollary 2.2) *Let n be a positive integer, $0 < \beta \leq 2$, and let B, C be complex-valued functions defined in $\mathbb{U}$, with*

$$|\text{Im } C(z)| \leq \left(n + \frac{2-\beta}{2+\beta}\right) \text{Re } B(z). \tag{3.1}$$

If $p \in \mathcal{H}_\beta[1,n]$ satisfies

$$\text{Re } \left(B(z)zp'(z) + C(z)p(z)\right) > 0, \tag{3.2}$$

then $\text{Re } p(z) > 0$.

By appealing to Lemma 3.1, the following result describes the integral operator **I** that satisfies the condition

$$\text{Re } f(z) > 0 \qquad \Rightarrow \qquad \text{Re } \mathbf{I}[f](z) > 0,$$

where $\mathbf{I}$ is defined on the class $\mathcal{P} \subset \mathcal{H}$.

Theorem 3.1. *Let n be a positive integer, $0 < \beta \leq 2$, and $\gamma \neq 0$ in $\mathbb{C}$ with $\text{Re } \gamma > -n$. Suppose $\varphi, \phi \in \mathcal{D}_{n,-\beta}$, and satisfy*

$$\left| \text{Im} \left(\frac{\gamma\phi(z) + z\phi'(z)}{\gamma\varphi(z)} \right) \right| \leq \left(n + \frac{2-\beta}{2+\beta} \right) \text{Re } \left(\frac{\phi(z)}{\gamma\varphi(z)} \right). \tag{3.3}$$

Let $\mathbf{I} : \mathcal{P}_{n,\beta} \to \mathcal{P}_{n,\beta}$ be defined by

$$\mathbf{I}[f](z) = \frac{\gamma}{z^\gamma \phi(z)} \int_0^z f(t)\varphi(t)t^{\gamma-1}\, dt. \tag{3.4}$$

Then $\mathbf{I}[\mathcal{P}_{n,\beta}] \subset \mathcal{P}_{n,\beta}$.

Proof. Write the integral operator in (3.4) as

$$F(z) = \mathbf{I}[f](z) = \frac{\gamma}{z^\gamma \phi(z)} \int_0^z g(t)t^{\gamma-1}\, dt := \frac{G(z)}{\phi(z)},$$

where $g(t) = f(t)\varphi(t) = 1 + \sum_{m=n+1}^{\infty} a_m t^m$. It is evident that

$$\begin{aligned}
G(z) &= \frac{\gamma}{z^\gamma} \int_0^z g(t)t^{\gamma-1}\, dt \\
&= \frac{\gamma}{z^\gamma} \int_0^z \left(1 + \sum_{m=n+1}^{\infty} a_m t^m \right) t^{\gamma-1}\, dt \\
&= \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1}\, dt \; + \; \frac{\gamma}{z^\gamma} \left(\sum_{m=n+1}^{\infty} a_m \int_0^z t^{m+\gamma-1}\, dt \right) \\
&= \frac{\gamma}{z^\gamma} \left(\frac{z^\gamma}{\gamma} \right) \; + \; \frac{\gamma}{z^\gamma} \sum_{m=n+1}^{\infty} \frac{a_m}{m+\gamma} z^{m+\gamma} \\
&= 1 + \sum_{m=n+1}^{\infty} \frac{\gamma a_m}{m+\gamma} z^m
\end{aligned}$$

lies in $\mathcal{H}[1, n+1]$. Thus

$$
\begin{aligned}
F(z) &= \frac{G(z)}{\phi(z)} \\
&= \left(1+\sum_{m=n+1}^{\infty}\frac{\gamma a_m}{m+\gamma}z^m\right)\left(1+\sum_{m=n}^{\infty}\phi_m z^m\right)^{-1} \\
&= \left(1+\sum_{m=n+1}^{\infty}\frac{\gamma a_m}{m+\gamma}z^m\right)\left(1-\sum_{m=n}^{\infty}\phi_m z^m+\left(\sum_{m=n}^{\infty}\phi_m z^m\right)^2-\cdots\right) \\
&= \left(1+\frac{\gamma a_{n+1}}{n+1+\gamma}z^{n+1}+\frac{\gamma a_{n+2}}{n+2+\gamma}z^{n+2}+\frac{\gamma a_{n+3}}{n+3+\gamma}z^{n+3}+\cdots\right) \\
&\qquad \times\Big[1-\big(-\beta z^n+\phi_{n+1}z^{n+1}+\phi_{n+2}z^{n+2}+\cdots\big) \\
&\qquad\qquad +\big(\beta^2 z^{2n}-2\beta\phi_{n+1}z^{2n+1}+(\phi_{n+1}^2-2\beta\phi_{n+2})z^{2n+2}+\cdots\big)-\cdots\Big] \\
&= 1+\beta z^n+\left(\frac{\gamma a_{n+1}}{n+1+\gamma}-\phi_{n+1}\right)z^{n+1}+\cdots,
\end{aligned}
$$

so that F is well-defined and $F\in\mathcal{P}_{n,\beta}$.

By letting

$$
B(z)=\frac{\phi(z)}{\gamma\varphi(z)}, \qquad \text{and} \qquad C(z)=\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\varphi(z)}, \tag{3.5}
$$

it is clear that condition (3.3) is equivalent to condition (3.1) in Lemma 3.1. Differentiating F with respect to z yields

$$
\begin{aligned}
F'(z) &= \frac{\partial}{\partial z}\left[\frac{\gamma}{z^\gamma\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right] \\
&= \frac{\gamma}{z^\gamma\phi(z)}\left[\frac{\partial}{\partial z}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right]+\left[\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right]\left[\frac{\partial}{\partial z}\left(\frac{\gamma}{z^\gamma\phi(z)}\right)\right] \\
&= \frac{\gamma}{z^\gamma\phi(z)}\left[f(z)\varphi(t)z^{\gamma-1}\right] \\
&\qquad +\left[\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right]\left[\frac{-\gamma\big(\gamma z^{\gamma-1}\phi(z)+z^\gamma\phi'(z)\big)}{\big(z^\gamma\phi(z)\big)^2}\right] \\
&= \frac{\gamma f(z)\varphi(z)}{z\phi(z)}-\left(\frac{\gamma\phi(z)+z\phi'(z)}{z\phi(z)}\right)\frac{\gamma}{z^\gamma\phi(z)}\int_0^z f(t)\varphi(z)t^{\gamma-1}\,dt,
\end{aligned}
$$

that is,

$$F'(z) = \frac{\gamma f(z)\varphi(z)}{z\phi(z)} - \left(\frac{\gamma\phi(z)+z\phi'(z)}{z\phi(z)}\right)F(z).$$

Multiplying by $z\phi(z)/\gamma\varphi(z)$ to the above equality gives

$$\left(\frac{\phi(z)}{\gamma\varphi(z)}\right)zF'(z) = f(z) - \left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\varphi(z)}\right)F(z),$$

which in turn implies

$$\left(\frac{\phi(z)}{\gamma\varphi(z)}\right)zF'(z) + \left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\varphi(z)}\right)F(z) = f(z).$$

In view of (3.5), it is evident that

$$B(z)zF'(z)+C(z)F(z) = f(z).$$

Since

$$\mathrm{Re}\ \big(B(z)zF'(z)+C(z)F(z)\big) = \mathrm{Re}\ f(z) > 0,$$

by Lemma 3.1, Re $F(z) > 0$ or Re $\mathbf{I}[f](z) > 0$, and hence, $\mathbf{I}[\mathcal{P}_{n,\beta}] \subset \mathcal{P}_{n,\beta}$. □

Taking $\beta = 2$ in Theorem 3.1, the following result of Miller and Mocanu [35] is readily obtained.

Corrollary 3.1. [35, Theorem 4.2a, p. 202] *Let* $\gamma \neq 0$ *in* $\mathbb{C}$ *with* Re $\gamma \geq 0$*, and* n *be a positive integer. Let* $\varphi, \phi \in \mathcal{D}_n$ *and suppose that*

$$\left|\mathrm{Im}\left(\frac{\gamma\phi(z)+z\phi'(z)}{\gamma\varphi(z)}\right)\right| \leq n\,\mathrm{Re}\ \left(\frac{\phi(z)}{\gamma\varphi(z)}\right).$$

If the integral operator $\mathbf{I}$ *be defined by*

$$\mathbf{I}[f](z) = \frac{\gamma}{z^{\gamma}\phi(z)}\int_0^z f(t)t^{\gamma-1}\varphi(t)\ dt,$$

then $\mathbf{I}[\mathcal{P}_n] \subset \mathcal{P}_n$.

3.3 Integral Operators Preserving Bounded Functions

In this section, denote by $\mathcal{HB}(M)[0,1]$ the class of analytic functions f satisfying $|f(z)| < M$ for $M > 0$ given by

$$\mathcal{HB}(M)[0,1] := \{f \in \mathcal{H}[0,1] : |f(z)| < M,\ M > 0, \quad \text{for} \quad z \in \mathbb{U}\}.$$

For $\beta > 0$, consider the subclass $\mathcal{HB}_\beta(M)[0,1]$ of $\mathcal{HB}(M)[0,1]$, consisting of all analytic functions f in $\mathbb{U}$ for which $|f(z)| < M,\ M > 0:$

$$\mathcal{HB}_\beta(M)[0,1] := \{f \in \mathcal{H}_\beta[0,1] : f(z) = \beta z + f_2 z^2 + f_3 z^3 + \cdots,\ |f(z)| < M\}.$$

The following lemma which is the Corollary 2.5 in Chapter 2 is required later.

Lemma 3.2. (Corollary 2.5) *Let $M > 0$ and $B, C : \mathbb{U} \to \mathbb{C}$ with $B(z) \neq 0$. If $p \in \mathcal{H}_\beta[0,1]$, $0 < \beta \leq N$, and*

$$N = \sup_{|z|<1} \left\{ \frac{M}{|B(z)| \cdot \left| \mathrm{Im}\left(\frac{C(z)}{B(z)}\right) \right|} \right\} < +\infty, \tag{3.6}$$

then

$$|B(z)zp'(z) + C(z)p(z)| < M \qquad \Rightarrow \qquad |p(z)| < N. \tag{3.7}$$

By using Lemma 3.2 and following the same approach as the previous section leads to the following result describing the linear integral operator $\mathbf{I}$ defined on the subclass $\mathcal{HB}(M) \subset \mathcal{H}$ satisfying

$$|f(z)| < M \qquad \Rightarrow \qquad |\mathbf{I}[f](z)| < N$$

with the bound $N > 0$ dependent on $\mathbf{I}$ and $M > 0$.

Theorem 3.2. *Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}\ \gamma > -1$. Further, let $\varphi, \phi \in \mathcal{D}$ and let $f \in \mathcal{HB}_\beta(M)[0,1]$, $M > 0$. Suppose $0 < \beta \leq N$ and*

$$N = \sup_{|z|<1} \left\{ \frac{M|(1+\gamma)\varphi(z)|}{|\phi(z)| \left| \mathrm{Im}\, \left(\gamma + \frac{z\phi'(z)}{\phi(z)} \right) \right|} \right\} < +\infty. \tag{3.8}$$

Let $\mathbf{I} : \mathcal{HB}_\beta(M)[0,1] \to \mathcal{HB}_\beta(N)[0,1]$ *be defined by*

$$\mathbf{I}[f](z) = \frac{1+\gamma}{z^\gamma \phi(z)} \int_0^z f(t)\varphi(t)t^{\gamma-1}\, dt. \tag{3.9}$$

Then $\mathbf{I}[\mathcal{HB}_\beta(M)[0,1]] \subset \mathcal{HB}_\beta(N)[0,1]$.

Proof. Express the integral operator in (3.9) as

$$F(z) = \mathbf{I}[f](z) = \frac{1+\gamma}{z^\gamma \phi(z)} \int_0^z h(t)t^{\gamma-1} dt := \frac{H(z)}{\phi(z)},$$

where $h(t) = f(t)\varphi(t) = \sum_{m=1}^{\infty} a_m t^m$. It follows that

$$\begin{aligned}
H(z) &= \frac{1+\gamma}{z^\gamma} \int_0^z h(t)t^{\gamma-1}\, dt \\
&= \frac{1+\gamma}{z^\gamma} \int_0^z \left(\sum_{m=1}^{\infty} a_m t^m \right) t^{\gamma-1}\, dt \\
&= \frac{1+\gamma}{z^\gamma} \sum_{m=1}^{\infty} a_m \int_0^z t^{m+\gamma-1}\, dt \\
&= \frac{1+\gamma}{z^\gamma} \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^{m+\gamma} \\
&= (1+\gamma) \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^m
\end{aligned}$$

lies in $\mathcal{H}[0,1]$. Thus

$$\begin{aligned}
F(z) &= \frac{H(z)}{\phi(z)} \\
&= \left((1+\gamma) \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^m \right) \left(1 + \sum_{n=1}^{\infty} \phi_n z^n \right)^{-1} \\
&= \left((1+\gamma) \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^m \right) \left(1 - \sum_{n=1}^{\infty} \phi_n z^n + \left(\sum_{n=1}^{\infty} \phi_n z^n \right)^2 - \cdots \right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(1+\gamma)\beta}{1+\gamma}z + \frac{(1+\gamma)a_2}{2+\gamma}z^2 + \frac{(1+\gamma)a_3}{3+\gamma}z^3 + \cdots\right) \\
&\quad \times \Bigg[1 - \left(\phi_1 z + \phi_2 z^2 + \phi_3 z^3 + \cdots\right) \\
&\qquad\quad + \left(\phi_1^2 z^2 + 2\phi_1\phi_2 z^3 + (2\phi_1\phi_3 + \phi_2^2)z^4 + \cdots\right) - \cdots\Bigg] \\
&= \beta z + \left(\frac{(1+\gamma)a_2}{2+\gamma} - \beta\phi_1\right)z^2 \\
&\qquad + \left(\frac{(1+\gamma)a_3}{3+\gamma} - \frac{(1+\gamma)a_2\phi_1}{2+\gamma} + \beta(\phi_1^2 - \phi_2)\right)z^3 + \cdots .
\end{aligned}
$$

The restriction on the terms γ, ϕ, φ and f implies that F is well-defined and $F \in \mathcal{HB}_\beta(N)[0,1]$.

By taking

$$
B(z) = \frac{\phi(z)}{(1+\gamma)\varphi(z)}, \qquad \text{and} \qquad C(z) = \frac{\gamma\phi(z) + z\phi'(z)}{(1+\gamma)\varphi(z)}, \tag{3.10}
$$

it is clear that condition (3.8) is equivalent to condition (3.6) in Lemma 3.2. Differentiating F with respect to z yields

$$
\begin{aligned}
F(z) &= \frac{\partial}{\partial z}\left[\frac{1+\gamma}{z^\gamma\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right] \\
&= \frac{1+\gamma}{z^\gamma\phi(z)}\left[\frac{\partial}{\partial z}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right] + \left[\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right]\left[\frac{\partial}{\partial z}\left(\frac{1+\gamma}{z^\gamma\phi(z)}\right)\right] \\
&= \frac{1+\gamma}{z^\gamma\phi(z)}\left[f(z)\varphi(z)z^{\gamma-1}\right] \\
&\qquad + \left[\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt\right]\left[\frac{-(1+\gamma)\big(\gamma z^{\gamma-1}\phi(z) + z^\gamma\phi'(z)\big)}{\big(z^\gamma\phi(z)\big)^2}\right] \\
&= \frac{(1+\gamma)f(z)\varphi(z)}{z\phi(z)} - \left(\frac{\gamma\phi(z) + z\phi'(z)}{z\phi(z)}\right)\frac{1+\gamma}{z^\gamma\phi(z)}\int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt,
\end{aligned}
$$

that is,

$$
F'(z) = \frac{(1+\gamma)f(z)\varphi(z)}{z\phi(z)} - \left(\frac{\gamma\phi(z) + z\phi'(z)}{z\phi(z)}\right)F(z).
$$

Multiplying by $z\phi(z)/[(1+\gamma)\varphi(z)]$ to the last equality gives

$$\left[\frac{\phi(z)}{(1+\gamma)\varphi(z)}\right] zF'(z) = f(z) - \left(\frac{\gamma\phi(z)+z\phi'(z)}{(1+\gamma)\varphi(z)}\right) F(z),$$

which upon simplification leads to

$$\left(\frac{\phi(z)}{(1+\gamma)\varphi(z)}\right) zF'(z) + \left(\frac{\gamma\phi(z)+z\phi'(z)}{(1+\gamma)\varphi(z)}\right) F(z) = f(z).$$

In view of (3.10), it is evident that

$$B(z)zF'(z) + C(z)F(z) = f(z).$$

Since

$$|B(z)zF'(z) + C(z)F(z)| = |f(z)| < M,$$

by Lemma 3.2, then $|F(z)| < N$, or $|\mathbf{I}[f](z)| < N$, and hence, $\mathbf{I}[\mathcal{HB}_\beta(M)[0,1]] \subset \mathcal{HB}_\beta(N)[0,1]$. $\square$

The following example illustrates the similar form of Theorem 3.2.

Example 3.1. *Let* $\gamma = x+iy$ *satisfies* $y > 0$ *and* $0 < 1+x < |\lambda| < y$. *For* $\varphi(z) = \phi(z) = e^{\lambda z}$, *consider*

$$\mathbf{I}(z) = \frac{1+\gamma}{z^\gamma e^{\lambda z}} \int_0^z f(t) e^{\lambda t} t^{\gamma-1}\, dt,$$

and

$$N = \sup_{|z|<1} \left\{ \frac{M|1+\gamma||e^{\lambda z}|}{|e^{\lambda z}|\,|\mathrm{Im}\,(\gamma+\lambda z)|} \right\} = \sup_{|z|<1} \left\{ \frac{M|1+\gamma|}{|\mathrm{Im}\,(\gamma+\lambda z)|} \right\} = \frac{M|1+\gamma|}{y-|\lambda|} < \infty.$$

Hence,

$$|f(z)| \leq M \qquad \Rightarrow \qquad \left| \frac{1}{z^\gamma e^{\lambda z}} \int_0^z f(t) e^{\lambda t} t^{\gamma-1}\, dt \right| \leq \frac{M}{y-|\lambda|}.$$

3.4 Integral Operators Preserving Convex Functions

In the present section, denote by $\mathcal{H}_C[0,1]$ the class of analytic functions f which are convex in $\mathbb{U}$ given by

$$\mathcal{H}_C[0,1] := \{f \in \mathcal{H}[0,1] : f \text{ is convex} \quad \text{for} \quad z \in \mathbb{U}\}.$$

For $\beta > 0$, consider the subclass $\mathcal{H}_{C\beta}[0,1]$ of $\mathcal{H}_C[0,1]$, consisting of all analytic functions f in $\mathbb{U}$ which are convex:

$$\mathcal{H}_{C\beta}[0,1] := \{f \in \mathcal{H}_\beta[0,1] : f(z) = \beta z + f_2 z^2 + f_3 z^3 + \cdots,\ f \text{ is convex}\}.$$

To prove the main result in this section, the following lemma which is Theorem 2.5 in Chapter 2 is required.

Lemma 3.3. (Theorem 2.5) *Let n be a positive integer. For $0 \leq \alpha < 1$, let h be a convex univalent function of order α in $\mathbb{U}$ with $h(0) = 0$. Further, let $0 < \beta \leq |h'(0)|$ and $k > 2^{2(1-\alpha)}/|h'(0)|$. Suppose that $A \geq 0$ and B, C, and D are analytic functions in $\mathbb{U}$ satisfying*

$$\begin{aligned} \operatorname{Re} B(z) \geq {} & \left[1 - \alpha\left(n + \frac{|h'(0)| - \beta}{|h'(0)| + \beta}\right)\right] A + \left(\frac{|h'(0)| + \beta}{(n+1)|h'(0)| + (n-1)\beta}\right) \\ & \times \left[\frac{1}{2\tau(\alpha)}|C(z) - 1| - \frac{1}{2\tau(\alpha)} \operatorname{Re}\, \big(C(z) - 1\big) + k|D(z)|\right], \end{aligned} \tag{3.11}$$

where

$$\tau(\alpha) := \begin{cases} \frac{2\alpha - 1}{2 - 2^{2(1-\alpha)}}, & if \quad \alpha \neq \frac{1}{2}, \\ \frac{1}{2\ln 2}, & if \quad \alpha = \frac{1}{2}. \end{cases}$$

If $p \in \mathcal{H}_\beta[0,n]$ satisfies the differential subordination

$$Az^2 p''(z) + B(z) z p'(z) + C(z) p(z) + D(z) \prec h(z), \tag{3.12}$$

then $p(z) \prec h(z)$.

The following result is obtained by appealing to Lemma 3.3.

Theorem 3.3. *Let* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\,\gamma > -1$. *Further, let* $\varphi, \phi \in \mathcal{D}$ *and let* $f \in \mathcal{H}_{C\beta}[0,1]$. *Let the integral operator* $\mathbf{I}$ *be defined by*

$$\mathbf{I}[f](z) = \frac{1+\gamma}{z^{\gamma}\phi(z)} \int_0^z f(t)t^{\gamma-1}\varphi(t)\,dt. \tag{3.13}$$

Let h *be a convex univalent function in* $\mathbb{U}$, *with* $h(0) = 0$ *and* $0 < \beta \le |h'(0)|$. *Suppose* $w \in \mathcal{H}[0,2]$ *and* B, C *are analytic functions in* $\mathbb{U}$ *satisfying*

$$\operatorname{Re}\,B(z) \ge \left(\frac{|h'(0)|+\beta}{2|h'(0)|}\right)\Big(|C(z)-1| - \operatorname{Re}\,(C(z)-1) + 4|w(z)|\Big), \tag{3.14}$$

where

$$B(z) = \frac{\phi(z)}{(1+\gamma)\varphi(z)} \qquad \text{and} \qquad C(z) = \frac{\gamma\phi(z) + z\phi'(z)}{(1+\gamma)\varphi(z)}. \tag{3.15}$$

Let $I_w : \mathcal{H}_{C\beta}[0,1] \to \mathcal{H}_{C\beta}[0,1]$ *be defined by*

$$\mathbf{I}_w[f](z) := \mathbf{I}[f + f'(0)w](z), \tag{3.16}$$

then $\mathbf{I}[\mathcal{H}_{C\beta}[0,1]] \subset \mathcal{H}_{C\beta}[0,1]$.

Proof. Write the integral operator in (3.16) as

$$F(z) = \mathbf{I}_w[f](z) = \mathbf{I}[f + f'(0)w](z) = \frac{1+\gamma}{z^{\gamma}\phi(z)} \int_0^z q(t)t^{\gamma-1}\,dt := \frac{Q(z)}{\phi(z)},$$

where the function $q(t) = [f(t) + f'(0)w(t)]\varphi(t) = \sum_{m=1}^{\infty} a_m t^m$. Evidently,

$$\begin{aligned}
Q(z) &= \frac{1+\gamma}{z^{\gamma}} \int_0^z q(t)t^{\gamma-1}\,dt \\
&= \frac{1+\gamma}{z^{\gamma}} \int_0^z \left(\sum_{m=1}^{\infty} a_m t^m\right) t^{\gamma-1}\,dt \\
&= \frac{1+\gamma}{z^{\gamma}} \sum_{m=1}^{\infty} a_m \int_0^z t^{m+\gamma-1}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1+\gamma}{z^\gamma} \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^{m+\gamma} \\
&= (1+\gamma) \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^m
\end{aligned}
$$

lies in $\mathcal{H}[0,1]$. Thus

$$
\begin{aligned}
F(z) &= \frac{Q(z)}{\phi(z)} \\
&= \left((1+\gamma) \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^m \right) \left(1 + \sum_{n=1}^{\infty} \phi_n z^n \right)^{-1} \\
&= \left((1+\gamma) \sum_{m=1}^{\infty} \frac{a_m}{m+\gamma} z^m \right) \left(1 - \sum_{n=1}^{\infty} \phi_n z^n + \left(\sum_{n=1}^{\infty} \phi_n z^n \right)^2 - \cdots \right) \\
&= \left(\frac{(1+\gamma)\beta}{1+\gamma} z + \frac{(1+\gamma)a_2}{2+\gamma} z^2 + \frac{(1+\gamma)a_3}{3+\gamma} z^3 + \cdots \right) \\
&\qquad \times \Bigg[1 - \left(\phi_1 z + \phi_2 z^2 + \phi_3 z^3 + \cdots \right) \\
&\qquad\qquad + \left(\phi_1^2 z^2 + 2\phi_1\phi_2 z^3 + (2\phi_1\phi_3 + \phi_2^2) z^4 + \cdots \right) - \cdots \Bigg] \\
&= \left(\beta z + \frac{(1+\gamma)a_2}{2+\gamma} z^2 + \frac{(1+\gamma)a_3}{3+\gamma} z^3 + \cdots \right) \\
&\qquad \times \left(1 - \phi_1 z + (\phi_1^2 - \phi_2) z^2 + (2\phi_1\phi_2 - \phi_3) z^3 + \cdots \right) \\
&= \beta z + \left(\frac{(1+\gamma)a_2}{2+\gamma} - \beta\phi_1 \right) z^2 \\
&\qquad + \left(\frac{(1+\gamma)a_3}{3+\gamma} - \frac{(1+\gamma)a_2\phi_1}{2+\gamma} + \beta(\phi_1^2 - \phi_2) \right) z^3 + \cdots .
\end{aligned}
$$

The restriction on the terms γ, φ, ϕ and w imply that F is well-defined and $F \in \mathcal{H}_{C\beta}[0,1]$.

Differentiating F with respect to z yields

$$
\begin{aligned}
F'(z) &= \frac{\partial}{\partial z} \left[\frac{1+\gamma}{z^\gamma \phi(z)} \int_0^z [f(t) + f'(0)w(t)] \varphi(t) t^{\gamma-1}\, dt \right] \\
&= \frac{1+\gamma}{z^\gamma \phi(z)} \left[\frac{\partial}{\partial z} \int_0^z [f(t) + f'(0)w(t)] \varphi(t) t^{\gamma-1}\, dt \right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left[\int_0^z [f(t)+f'(0)w(t)]\varphi(t)t^{\gamma-1}\,dt\right]\left[\frac{\partial}{\partial z}\left(\frac{1+\gamma}{z^\gamma\phi(z)}\right)\right] \\
&= \frac{1+\gamma}{z^\gamma\phi(z)}\left[[f(z)+f'(0)w(z)]\varphi(t)z^{\gamma-1}\right] \\
&\quad + \left[\int_0^z [f(t)+f'(0)w(t)]\varphi(t)t^{\gamma-1}\,dt\right]\left[\frac{-(1+\gamma)\big(\gamma z^{\gamma-1}\phi(z)+z^\gamma\phi'(z)\big)}{\big(z^\gamma\phi(z)\big)^2}\right] \\
&= \frac{(1+\gamma)[f(z)+f'(0)w(z)]\varphi(z)}{z\phi(z)} \\
&\qquad - \left(\frac{\gamma\phi(z)+z\phi'(z)}{z\phi(z)}\right)\frac{1+\gamma}{z^\gamma\phi(z)}\int_0^z [f(t)+f'(0)w(t)]\varphi(t)t^{\gamma-1}\,dt,
\end{aligned}
$$

that is,

$$
F'(z) = \frac{(1+\gamma)[f(z)+f'(0)w(z)]\varphi(z)}{z\phi(z)} - \left(\frac{\gamma\phi(z)+z\phi'(z)}{z\phi(z)}\right)F(z).
$$

Multiplying by $z\phi(z)/[(1+\gamma)\varphi(z)]$ to the above equality gives

$$
\left(\frac{\phi(z)}{(1+\gamma)\varphi(z)}\right)zF'(z) = f(z)+f'(0)w(z) - \left(\frac{\gamma\phi(z)+z\phi'(z)}{(1+\gamma)\varphi(z)}\right)F(z),
$$

which on simplification gives

$$
\left(\frac{\phi(z)}{(1+\gamma)\varphi(z)}\right)zF'(z) + \left(\frac{\gamma\phi(z)+z\phi'(z)}{(1+\gamma)\varphi(z)}\right)F(z) - f'(0)w(z) = f(z).
$$

In view of (3.15) and let $D(z) = -f'(0)w(z)$, it is evident that

$$
B(z)zF'(z)+C(z)F(z)+D(z) = f(z).
$$

The aim to prove this theorem is to make use of Lemma 3.3. Let

$$
B(z)zF'(z)+C(z)F(z)+D(z) = f(z) \prec h(z).
$$

Since $f(z) \prec h(z)$, it implies that

$$|f'(0)| \leq |h'(0)| \qquad \text{or} \qquad \frac{1}{|f'(0)|} \geq \frac{1}{|h'(z)|}.$$

The above fact shows that

$$4|w(z)| = 4\frac{|D(z)|}{|f'(0)|} \geq 4\frac{|D(z)|}{|h'(0)|}.$$

since $D(z) = -f'(0)w(z)$. It is clear that condition (3.14) implies condition (3.11) in Lemma 3.3 when $A = 0$, $\alpha = 0$ and $n = 1$. By Lemma 3.3, $F(z) \prec h(z)$, or $\mathbf{I}_w[f](z) \prec h(z)$, and hence $\mathbf{I}[\mathcal{H}_{C\beta}[0,1]] \subset \mathcal{H}_{C\beta}[0,1]$. $\square$

CHAPTER 4

SUBORDINATION OF THE SCHWARZIAN DERIVATIVE

4.1 Introduction

Recall that $\mathcal{A}$ be the subclass of $\mathcal{H}[0,1]$ consisting of normalized analytic functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in $\mathbb{U}$. The subclasses of $\mathcal{A}$ consisting of starlike and convex functions in $\mathbb{U}$ are denoted by $\mathcal{ST}$ and $\mathcal{CV}$, respectively.

The Schwarzian derivative of a locally univalent analytic function f in $\mathbb{U}$ is defined by

$$\{f,z\} := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

In 1978, Miller and Mocanu [33] found conditions on $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that

$$\operatorname{Re}\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right)\right\} > 0$$

implies $f \in \mathcal{ST}$. As applications, if $f \in \mathcal{A}$ satisfies either

$$\operatorname{Re}\left\{\alpha\left(\frac{zf'(z)}{f(z)}\right) + \delta\left(1+\frac{zf''(z)}{f'(z)}\right) + \left(\frac{zf'(z)}{f(z)}\right)z^2\{f,z\}\right\} > 0, \qquad (\alpha,\delta \in \mathbb{R}),$$

or

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} + \left(1+\frac{zf''(z)}{f'(z)} + z^2\{f,z\}\right)\right\} > -\frac{1}{2},$$

then $f \in \mathcal{ST}$. Later, Owa and Obradovic [46, Corollary 2, p. 490] proved that if $f \in \mathcal{A}$ and

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\} > -\frac{1}{2},$$

then $f \in \mathcal{ST}$.

Miller and Mocanu [33] also found conditions on $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ such that

$$\operatorname{Re}\left\{\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right)\right\} > 0$$

implies $f \in \mathcal{CV}$. They proved that $f \in \mathcal{CV}$ if $f \in \mathcal{A}$ satisfies one of the following conditions:

(i) $\operatorname{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right) + \alpha z^2\{f,z\}\right\} > 0, \qquad (\operatorname{Re}\ \alpha \geq 0)$,

(ii) $\operatorname{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)^2 + z^2\{f,z\}\right\} > 0$,

(iii) $\operatorname{Re}\left\{\left(1+\frac{zf''(z)}{f'(z)}\right) e^{z^2\{f,z\}}\right\} > 0$.

In fact, Owa and Obradovic [46, Corollary 3, p. 490] also proved that $f \in \mathcal{CV}$ if $f \in \mathcal{A}$ and

$$\operatorname{Re}\left\{z^2\{f,z\} + \frac{1}{2}\left(1+\frac{zf''(z)}{f'(z)}\right)^2\right\} > 0.$$

This chapter deals with the class $\mathcal{H}_\beta[a,n]$ of all analytic functions p of the form

$$p(z) = a + \beta z^n + p_{n+1}z^{n+1} + \cdots,$$

where the fixed coefficient β is a non-negative real number. Denote by $\mathcal{A}_{n,b}$ the class of all normalized analytic functions $f \in \mathcal{A}_n$ of the form

$$f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots,$$

where b is a fixed non-negative real number. Write $\mathcal{A}_{1,b} = \mathcal{A}_b$.

Recently, Miller and Mocanu [33] obtained sufficient conditions for starlikeness and convexity of functions $f \in \mathcal{A}$ in term of the Schwarzian derivative. Ali et al. [3] have used the result of Miller and Mocanu [33] and obtained sufficient conditions for functions $f \in \mathcal{A}$ involving the Schwarzian derivative to satisfy either

$$q_1 \prec \frac{zf'(z)}{f(z)} \prec q_2 \qquad \text{or} \qquad q_1 \prec 1 + \frac{zf''(z)}{f'(z)} \prec q_2,$$

where q_1 is analytic and q_2 is analytic univalent in $\mathbb{U}$.

In this chapter, subordination is investigated on a class of β-admissible functions. The aim of this chapter is to obtain sufficient conditions in term of Schwarzian derivative to ensure functions $f \in \mathcal{A}_b$ to satisfy either

$$\frac{zf'(z)}{f(z)} \prec q(z) \qquad \text{or} \qquad 1 + \frac{zf''(z)}{f'(z)} \prec q(z),$$

where q is analytic univalent in $\mathbb{U}$. In Section 4.2, a class of β-admissible functions related to starlikeness satisfying the subordination implication involving the Schwarzian derivative is determined. Sufficient conditions in terms of the Schwarzian derivative that implies starlikeness of functions $f \in \mathcal{A}_b$ are obtained. These results extend earlier works by [3, 35].

Section 4.3 is devoted to finding sufficient conditions in terms of the Schwarzian derivative that implies convexity of functions $f \in \mathcal{A}_b$. At the beginning of this section, a class of β-admissible functions related to convexity satisfying the subordination implication involving the Schwarzian derivative is obtained. These results also extend earlier works by [3, 35].

Again, the following results obtained by Ali et al. [7] are required in the sequel. Let Q denote the set of functions q that are analytic and univalent on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$. The subclass of Q for which $q(0) = a$ is denoted by $Q(a)$ with $Q(1) := Q_1$.

Definition 4.1. [7, Definition 3.1, p. 616] *Let Ω be a domain in $\mathbb{C}$, $q \in Q$, $\beta \in \mathbb{R}$ with $0 < \beta \leq |q'(0)|$ and let n be a positive integer. The class $\Psi_{n,\beta}(\Omega, q)$ consists of β-admissible functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfying the β-admissibility condition*

$$\psi(r,s,t;z) \notin \Omega$$

whenever

$$\begin{aligned} r = q(\zeta), \qquad s &= m\zeta q'(\zeta), \quad \text{and} \\ \operatorname{Re}\left(\frac{t}{s}+1\right) &\geq m\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right), \end{aligned}$$

where $|\zeta| = 1$, $q(\zeta)$ is finite and

$$m \geq n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}.$$

The class $\Psi_{1,\beta}(\Omega, q)$ is denoted by $\Psi_\beta(\Omega, q)$.

Lemma 4.1. [7, Theorem 3.1, p. 617] *Let $q(0) = a$, $\psi \in \Psi_{n,\beta}(\Omega, q)$ with associated domain* D*, and $\beta \in \mathbb{R}$ with $0 < \beta \leq |q'(0)|$. Let $f \in \mathcal{H}_\beta[a,n]$. If $(p(z), zp'(z)z^2p''(z); z) \in$* D *and*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \qquad (z \in \mathbb{U}),$$

then $p(z) \prec q(z)$.

4.2 Starlikeness and Subordination of The Schwarzian Derivative

The following class of β-admissible functions related to starlikeness is introduced.

Definition 4.2. (β-Admissibility Condition) *Let Ω be a domain in $\mathbb{C}$, $q \in Q_1$, $\beta \in \mathbb{R}$ with $0 < \beta \leq |q'(0)|$. The class $\Phi_{S,\beta}(\Omega, q)$ consists of β-admissible functions ϕ : $\mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfying the β-admissibility condition*

$$\phi(u,v,w;z) \notin \Omega$$

whenever

$$\begin{aligned} &u = q(\zeta), \qquad v = q(\zeta) + \frac{m\zeta q'(\zeta)}{q(\zeta)}, \qquad (q(\zeta) \neq 0), \\ &\operatorname{Re}\left(\frac{2w+u^2-1+3(v-u)^2}{2(v-u)}\right) \geq m \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right), \end{aligned} \tag{4.1}$$

$z \in \mathbb{U},\ \zeta \in \partial\mathbb{U} \setminus E(q)$ *and*

$$m \geq 1 + \frac{|q'(0)| - \beta}{|q'(0)| + \beta} = \frac{2|q'(0)|}{|q'(0)| + \beta}. \tag{4.2}$$

The following is the main result in this section that make use of Definition 4.2.

Theorem 4.1. *Let* $f \in \mathcal{A}_b,\ 0 < \beta = b \leq |q'(0)|$, *with* $f(z)f'(z)/z \neq 0$. *If* $\phi \in \Phi_{S,\beta}(\Omega, q)$ *satisfies*

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{4.3}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

Proof. Define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{4.4}$$

Since $f \in \mathcal{A}_b$, a simple calculation shows that

$$\begin{aligned} p(z) &= \frac{z(1+\sum_{k=2}^{\infty} k a_k z^{k-1})}{z+\sum_{k=2}^{\infty} a_k z^k} \\ &= \frac{1+\sum_{k=2}^{\infty} k a_k z^{k-1}}{1+\sum_{k=2}^{\infty} a_k z^{k-1}} \\ &= \left(1+\sum_{k=2}^{\infty} k a_k z^{k-1}\right)\left(1-\sum_{k=2}^{\infty} a_k z^{k-1} + \left(\sum_{k=2}^{\infty} a_k z^{k-1}\right)^2 + \cdots\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(1+2bz+3a_3z^2+4a_4z^3+\cdots\right) \\
&\quad \times \left[1-\left(bz+a_3z^2+a_4z^3+\cdots\right)+\left(b^2z^2+2ba_3z^3+(2ba_4+a_3^2)z^4+\cdots\right)+\cdots\right] \\
&= \left(1+2bz+3a_3z^2+4a_4z^3+\cdots\right)\left(1-bz+(b^2-a_3)z^2+\cdots\right) \\
&= 1+bz+(2a_3-b^2)z^2+\cdots
\end{aligned}
$$

is analytic in $\mathbb{U}$. Thus, $f \in \mathcal{H}_\beta[1,1]$ with $\beta = b$ and $p(0)=1$. By taking the logarithmic differentiation on (4.4) gives

$$\frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + 1 - p(z),$$

and so

$$\frac{zf''(z)}{f'(z)} + 1 = p(z) + \frac{zp'(z)}{p(z)}. \tag{4.5}$$

It is evident from the definition of $\{f,z\}$ that

$$z^2\{f,z\} = z^2\left[\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right].$$

By using the value of $f''(z)/f'(z)$ from (4.5), the above equality is equivalent to

$$z^2\{f,z\} = z^2\left[\left(\frac{p(z)-1}{z} + \frac{p'(z)}{p(z)}\right)' - \frac{1}{2}\left(\frac{p(z)-1}{z} + \frac{p'(z)}{p(z)}\right)^2\right].$$

A further computation shows that

$$
\begin{aligned}
z^2\{f,z\} &= z^2\left(\frac{p'(z)}{z} - \frac{p(z)-1}{z^2} + \frac{p''(z)}{p(z)} - \left(\frac{p'(z)}{p(z)}\right)^2\right) \\
&\quad - \frac{z^2}{2}\left(\frac{p^2(z)-2p(z)+1}{z^2} + \frac{2p'(z)}{z} - \frac{2p'(z)}{zp(z)} + \left(\frac{p'(z)}{p(z)}\right)^2\right) \\
&= zp'(z) - p(z) + 1 + \frac{z^2p''(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2
\end{aligned}
$$

$$-\frac{p^2(z)}{2}+p(z)-\frac{1}{2}-zp'(z)+\frac{zp'(z)}{p(z)}-\frac{1}{2}\left(\frac{zp'(z)}{p(z)}\right)^2$$
$$=\frac{zp'(z)+z^2p''(z)}{p(z)}-\frac{3}{2}\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{1-p^2(z)}{2}. \tag{4.6}$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}^3$ by

$$\begin{aligned} u &= r, \\ v &= r+\frac{s}{r}, \\ w &= \frac{s+t}{r}-\frac{3s^2}{2r^2}+\frac{1-r^2}{2}, \end{aligned} \tag{4.7}$$

and the function $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ by

$$\begin{aligned} \psi(r,s,t;z) &= \phi(u,v,w;z) \\ &= \phi\left(r\ ,\ r+\frac{s}{r}\ ,\ \frac{s+t}{r}-\frac{3s^2}{2r^2}+\frac{1-r^2}{2}\ ;\ z\right). \end{aligned} \tag{4.8}$$

It follows from (4.4), (4.5), (4.6) and (4.8) that

$$\begin{aligned} &\psi(p(z),zp'(z),z^2p''(z);z) \\ &\quad=\phi\left(p(z)\ ,\ p(z)+\frac{zp'(z)}{p(z)}\ ,\ \frac{zp'(z)+z^2p''(z)}{p(z)}-\frac{3}{2}\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{1-p^2(z)}{2}\ ;\ z\right) \\ &\quad=\phi\left(\frac{zf'(z)}{f(z)}\ ,\ \frac{zf''(z)}{f'(z)}+1\ ,\ z^2\{f,z\}\ ;\ z\right). \end{aligned}$$

Hence, the condition (4.3) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z) \in \Omega.$$

In view of (4.7), it follows that

$$s = u(v-u),$$

and

$$t=\frac{2uw-2u(v-u)-u(1-u^2)+3u(v-u)^2}{2}.$$

This leads to

$$1+\frac{t}{s}=\frac{2w+u^2-1+3(v-u)^2}{2(v-u)}.$$

Thus the β-admissibility condition for $\phi\in\Phi_{S,\beta}(\Omega,q)$ in Definition 4.2 is equivalent to the β-admissibility condition for ψ as given in Definition 4.1. Hence, $\psi\in\Psi_\beta(\Omega,q)$ and by virtue of Lemma 4.1,

$$p(z)\prec q(z) \qquad \text{or} \qquad \frac{zf'(z)}{f(z)}\prec q(z).$$

□

In the special situation when $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{S,\beta}(h(\mathbb{U}),q)$ is written as $\Phi_{S,\beta}(h,q)$ and the following result is an immediate consequence of Theorem 4.1.

Theorem 4.2. *Let* $0<\beta=b\leq|q'(0)|$ *and* $\phi\in\Phi_{S,\beta}(h,q)$. *If* $f\in\mathcal{A}_b$ *with* $f(z)f'(z)/z\neq 0$ *and satisfying*

$$\phi\left(\frac{zf'(z)}{f(z)},1+\frac{zf''(z)}{f'(z)},z^2\{f,z\};z\right)\prec h(z),$$

then

$$\frac{zf'(z)}{f(z)}\prec q(z).$$

As an application, it is of interest to investigate Theorem 4.1 to the case of $q(\mathbb{U})$ being the right half-plane $q(\mathbb{U})=\{w:\operatorname{Re}\, w>0\}:=\triangle$.

Theorem 4.3. *Let* Ω *be a set in* $\mathbb{C}$ *and let the function* $\phi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ *satisfying the*

β-admissibility condition

$$\phi(i\rho, i\tau, \xi + i\eta; z) \notin \Omega,$$

for all $z \in \mathbb{U}$, $\rho, \tau, \xi, \eta \in \mathbb{R}$, and $0 < \beta \leq 2$ with

$$\rho\tau \geq \frac{1}{2+\beta}\left(2+(4+\beta)\rho^2\right), \qquad \textit{and} \qquad \rho\eta \geq 0. \tag{4.9}$$

If $f \in \mathcal{A}_b$, $0 \leq b \leq 2$, with $f(z)f'(z)/z \neq 0$ and to satisfy

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \in \Omega,$$

then $f \in \mathcal{ST}_b$.

Proof. Let the function

$$q(z) = \frac{1+z}{1-z},$$

then $q(0) = 1$ and $q \in Q_1$. For $\zeta \in \partial\mathbb{U} \setminus \{1\}$, it follows that

$$q(\zeta) = \frac{1+\zeta}{1-\zeta} = i\rho, \qquad \zeta q'(\zeta) = \frac{2\zeta}{(1-\zeta)^2} = -\frac{(1+\rho^2)}{2},$$

and

$$\zeta^2 q''(\zeta) = \frac{4\zeta^2}{(1-\zeta)^3} = \frac{(1+\rho^2)(1-i\rho)}{2}.$$

Note that

$$\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right) = \mathrm{Re}\left(\frac{2\zeta}{1-\zeta}+1\right) = \mathrm{Re}\,(i\rho) = 0 \tag{4.10}$$

where $\zeta = (i\rho-1)/(i\rho+1)$.

In this case, the β-admissibility condition for the class $\Phi_{S,\beta}(\Omega, (1+z)/(1-z))$ in Definition 4.2 is described by

$$
\begin{aligned}
u &= q(\zeta) \ := i\rho, \\
v &= q(\zeta) + \frac{m\zeta q'(\zeta)}{q(\zeta)} = i\left(\rho + \frac{m(1+\rho^2)}{2\rho}\right) \ := i\tau, \\
w &\ := \xi + i\eta,
\end{aligned} \tag{4.11}
$$

with

$$
\begin{aligned}
&\text{Re}\ \left(\frac{2w+u^2-1+3(v-u)^2}{2(v-u)}\right) \\
&= \text{Re}\ \left(\frac{2(\xi+i\eta)+(i\rho)^2-1+3(i\tau-i\rho)^2}{2(i\tau-i\rho)}\right) \\
&= \text{Re}\ \left(\left[2\xi+2i\eta-\rho^2-1+3\left(\frac{im(1+\rho^2)}{2\rho}\right)^2\right]\times\frac{\rho}{im(1+\rho^2)}\right) \\
&= \text{Re}\ \left(\frac{2i\rho\xi-2\rho\eta-i\rho^3-i\rho}{-m(1+\rho^2)}+\frac{3im(1+\rho^2)}{4\rho}\right) \\
&= \frac{2\rho\eta}{m(1+\rho^2)}.
\end{aligned} \tag{4.12}
$$

In view of (4.10) and (4.12), the condition (4.1) in Definition 4.2 reduces to

$$
\frac{2\rho\eta}{m(1+\rho^2)} \geq 0,
$$

which yields the desired condition $\rho\eta \geq 0$ as asserted in (4.9).

By using the value of τ in (4.11) leads to

$$
2\rho\tau = 2\rho^2 + m(1+\rho^2).
$$

Applying (4.2) in Definition 4.2 with $|q'(0)| = 2$ to the above equality yields

$$
2\rho\tau \geq 2\rho^2 + \frac{4(1+\rho^2)}{2+\beta},
$$

that is,

$$
\rho\tau \geq \frac{1}{2+\beta}\left(2+(4+\beta)\rho^2\right),
$$

which is the estimate (4.9). Thus the β-admissibility condition for functions in $\Phi_{S,\beta}(\Omega,(1+z)/(1-z))$ is equivalent to $\phi(i\rho,i\tau,\xi+i\eta;z)\notin\Omega$, whence $\phi\in\Phi_{S,\beta}(\Omega,(1+z)/(1-z))$. It follows from Theorem 4.1 that $f\in\mathcal{ST}_b$. $\square$

For $\beta=2$, Definition 4.2 reduces to Definition 2.1 in Ali et al. [3, p. 5]. In this case, the coefficient b satisfying the sharp bound for second coefficient when $f\in\mathcal{ST}$, that is, $|b|=|a_2|\leq 2$ and Theorem 4.3 coincides with the sufficient condition for $f\in\mathcal{A}$ obtained by Ali et al. [3].

Corollary 4.1. [3, Theorem 2.8, p. 9] *Let Ω be a set in $\mathbb{C}$ and let the function $\phi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ satisfying the admissibility condition*

$$\phi(i\rho,i\tau,\xi+i\eta;z)\notin\Omega,$$

for $z\in\mathbb{U}$, and for all real ρ,τ,ξ and η with

$$\rho\tau\geq\frac{1}{2}\left(1+3\rho^2\right),\qquad \rho\eta\geq 0.$$

Let $f\in\mathcal{A}$ with $f(z)f'(z)/z\neq 0$. If

$$\phi\left(\frac{zf'(z)}{f(z)},1+\frac{zf''(z)}{f'(z)},z^2\{f,z\};z\right)\in\Omega,\qquad z\in\mathbb{U},$$

then $f\in\mathcal{ST}$.

When $h(z)=(1+z)/(1-z)$, then $h(\mathbb{U})=q(\mathbb{U})$ or $\Omega=\triangle$ and the class of β-admissible functions $\Phi_{S,\beta}(h(\mathbb{U}),\triangle)$ is denoted by $\Phi_{S,\beta}(\triangle)$. Consequently, the following corollary is obtained by Theorem 4.3.

Corollary 4.2. *Let $\phi\in\Phi_{S,\beta}(\triangle)$ and satisfies*

$$\operatorname{Re}\{\phi(i\rho,i\tau,\zeta+i\eta;z)\}\leq 0,$$

when $\rho, \tau, \zeta, \eta \in \mathbb{R}$ *and* $0 < \beta \leq 2$ *with* $\rho\tau \geq [2+(4+\beta)\rho^2]/(2+\beta)$ *and* $\rho\eta \geq 0$. *If* $f \in \mathcal{A}_b$, $0 \leq b \leq 2$, *with* $f(z)f'(z)/z \neq 0$ *and satisfies*

$$\operatorname{Re}\left\{\phi\left(\frac{zf'(z)}{f(z)},\ \frac{zf''(z)}{f'(z)}+1,\ z^2\{f,z\};z\right)\right\} > 0, \qquad z \in \mathbb{U},$$

then $f \in \mathcal{ST}_b$.

In the case $\beta = 2$, the coefficient b will follow the same argument as before and Corollary 4.2 reduces to the following corollary.

Corrollary 4.3. [35, Theorem 4.6a, p. 244] *Let* $\phi : \mathbb{C}^3 \to \mathbb{C}$ *satisfy*

$$\operatorname{Re}\left\{\phi(i\rho, i\tau, \zeta + i\eta; z)\right\} \leq 0,$$

when $\rho, \tau, \zeta, \eta \in \mathbb{R}$, $\rho\tau \geq (1+3\rho^2)/2$, *and* $\rho\eta \geq 0$. *Let* $f \in \mathcal{A}$ *with* $f(z)f'(z)/z \neq 0$. *If*

$$\operatorname{Re}\left\{\phi\left(\frac{zf'(z)}{f(z)},\ \frac{zf''(z)}{f'(z)}+1,\ z^2\{f,z\};z\right)\right\} > 0, \qquad z \in \mathbb{U},$$

then $f \in \mathcal{ST}$.

4.3 Convexity and Subordination of The Schwarzian Derivative

The following class of β-admissible functions related to convexity is introduced.

Definition 4.3. (β-Admissibility Condition) *Let* Ω *be a domain in* $\mathbb{C}$, $q \in Q_1 \cap \mathcal{H}[1,1]$, $\beta \in \mathbb{R}$ *with* $0 < \beta \leq |q'(0)|$. *The class* $\Phi_{C,\beta}(\Omega, q)$ *consists of* β*-admissible functions* $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ *satisfying the* β*-admissibility condition*

$$\phi(u, v; z) \notin \Omega$$

whenever

$$u = q(\zeta), \qquad v = m\zeta q'(\zeta) + \frac{1-q^2(\zeta)}{2},$$

for $z \in \mathbb{U},\ \zeta \in \partial\mathbb{U} \setminus E(q)$ *and*

$$m \geq 1 + \frac{|q'(0)| - \beta}{|q'(0)| + \beta} = \frac{2|q'(0)|}{|q'(0)| + \beta}. \tag{4.13}$$

The following is the main result in this section that make use of Definition 4.3.

Theorem 4.4. *Let* $f \in \mathcal{A}_b,\ 0 < \beta = 2b \leq |q'(0)|$, *with* $f'(z) \neq 0$. *If* $\phi \in \Phi_{C,\beta}(\Omega, q)$ *satisfies*

$$\left\{ \phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f,z\}\right) : z \in \mathbb{U} \right\} \subset \Omega, \tag{4.14}$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z).$$

Proof. Define the function $p : \mathbb{U} \to \mathbb{C}$ by

$$p(z) = 1 + \frac{zf''(z)}{f'(z)}. \tag{4.15}$$

Since $f \in \mathcal{A}_b$, a simple calculation shows that

$$\begin{aligned}
p(z) &= 1 + \frac{z\left(\sum_{k=2}^{\infty} k(k-1)a_k z^{k-2}\right)}{1 + \sum_{k=2}^{\infty} k a_k z^{k-1}} \\
&= 1 + \left(\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}\right)\left(1 - \sum_{k=2}^{\infty} k a_k z^{k-1} + \left(\sum_{k=2}^{\infty} k a_k z^{k-1}\right)^2 + \cdots\right) \\
&= 1 + \left(2bz + 6a_3z^2 + 12a_4z^3 + \cdots\right)\left(1 - \left(2bz + 3a_3z^2 + 4a_4z^3 + \cdots\right)\right. \\
&\quad \left. + \left(4b^2z^2 + 12ba_3z^3 + (16ba_4 + 9a_3^2)z^4 + \cdots\right) + \cdots\right) \\
&= 1 + \left(2bz + 6a_3z^2 + 12a_4z^3 + \cdots\right)\left(1 - 2bz + (4b^2 - 3a_3)z^2 + (4a_4 + 12ba_3)z^3 + \cdots\right) \\
&= 1 + 2bz + (6a_3 - 4b^2)z^2 + (8b^3 - 18ba_3 + 12a_4)z^3 + \cdots
\end{aligned}$$

is analytic in $\mathbb{U}$. Thus $f \in \mathcal{H}_\beta[1,1]$ where $\beta = 2b$ and $p(0) = 1$.

It is evident from the definition of $\{f,z\}$ that

$$z^2\{f,z\} = z^2\left[\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right].$$

By substituting $f''(z)/f'(z)$ from (4.15) in the above equation leads to

$$z^2\{f,z\} = z^2\left[\left(\frac{p(z)-1}{z}\right)' - \frac{1}{2}\left(\frac{p(z)-1}{z}\right)^2\right].$$

A routine calculation shows that

$$\begin{aligned} z^2\{f,z\} &= z^2\left(\frac{zp'(z)-p(z)+1}{z^2}\right) - \frac{z^2}{2}\left(\frac{p^2(z)-2p(z)+1}{z^2}\right) \\ &= zp'(z) - p(z) + 1 - \frac{p^2(z)}{2} + p(z) - \frac{1}{2} \\ &= zp'(z) + \frac{1-p^2(z)}{2}. \end{aligned} \tag{4.16}$$

Define the transformation from $\mathbb{C}^2$ to $\mathbb{C}^2$ by

$$\begin{aligned} u &= r, \\ v &= s + \frac{1-r^2}{2}. \end{aligned}$$

If the function $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ is defined by

$$\begin{aligned} \psi(r,s;z) &= \phi(u\ ,\ v\ ;\ z) \\ &= \phi\left(r\ ,\ s + \frac{1-r^2}{2}\ ;\ z\right), \end{aligned} \tag{4.17}$$

then equations (4.15), (4.16) and (4.17) give

$$\begin{aligned}\psi(p(z), zp'(z); z) &= \phi\left(p(z)\ ,\ zp'(z) + \frac{1-p^2(z)}{2}\ ;\ z\right)\\ &= \phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}\ ;\ z\right).\end{aligned}$$

Hence, condition (4.14) becomes

$$\psi\left(p(z), zp'(z); z\right) \in \Omega.$$

Thus the β-admissibility condition for $\phi \in \Phi_{C,\beta}(\Omega, q)$ in Definition 4.3 is equivalent to the β-admissibility condition for ψ as given in Definition 4.1. Hence, $\psi \in \Psi_\beta(\Omega, q)$ and by virtue of Lemma 4.1,

$$p(z) \prec q(z) \quad \text{or} \quad 1+\frac{zf''(z)}{f'(z)} \prec q(z).$$

$\square$

In the case $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto $\Omega \neq \mathbb{C}$, the class $\Phi_{C,\beta}(h(\mathbb{U}), q)$ is simply denoted by $\Phi_{C,\beta}(h, q)$. The following result is established, which is stated without proof as a consequence of Theorem 4.4.

Theorem 4.5. *Let $0 < \beta = 2b \leq |q'(0)|$ and $\phi \in \Phi_{C,\beta}(h, q)$. If $f \in \mathcal{A}_b$ with $f'(z) \neq 0$ satisfies*

$$\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\}; z\right) \prec h(z),$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec q(z).$$

An interesting application of Theorem 4.4 is in the case of $q(\mathbb{U})$ being the right half-plane $q(\mathbb{U}) = \{w : \text{Re } w > 0\} := \triangle$.

Theorem 4.6. *Let Ω be a set in $\mathbb{C}$ and let the function $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfying the β-admissibility condition*

$$\phi(i\rho, \eta; z) \notin \Omega,$$

for all $z \in \mathbb{U}$, $\rho, \eta \in \mathbb{R}$, and $0 < \beta \leq 2$ with

$$\eta \leq \frac{\beta - 2}{2(2+\beta)}(1+\rho^2). \tag{4.18}$$

If $f \in \mathcal{A}_b$, $0 \leq b \leq 1$, with $f'(z) \neq 0$ and to satisfy

$$\phi\left(1 + \frac{zf''(z)}{f'(z)}, z^2\{f, z\}; z\right) \in \Omega, \qquad z \in \mathbb{U},$$

then $f \in \mathcal{CV}_b$.

Proof. Let the function

$$q(z) = \frac{1+z}{1-z},$$

then $q(0) = 1$ and $q \in Q_1 \cap \mathcal{H}[1,1]$. For $\zeta \in \partial\mathbb{U} \setminus \{1\}$, it follows that

$$q(\zeta) = \frac{1+\zeta}{1-\zeta} = i\rho,$$

and

$$\zeta q'(\zeta) = \frac{2\zeta}{(1-\zeta)^2} = -\frac{(1+\rho^2)}{2},$$

where $\zeta = (i\rho - 1)/(i\rho + 1)$.

In this case, the β-admissibility condition for the class of β-admissible functions $\Phi_{C,\beta}(\Omega, (1+z)/(1-z))$ in Definition 4.3 is described by

$$\begin{aligned} u &= q(\zeta) \ := i\rho, \\ v &= m\zeta q'(\zeta) + \frac{1 - q^2(\zeta)}{2} = \frac{(1-m)(1+\rho^2)}{2} \ := \eta. \end{aligned} \tag{4.19}$$

The value of η in (4.19) gives

$$2\eta = -m(1+\rho^2)+1+\rho^2.$$

By applying (4.13) in Definition 4.3 with $|q'(0)| = 2$ to the above equality yields

$$2\eta \leq -\frac{4}{2+\beta}(1+\rho^2)+1+\rho^2,$$

that is,

$$\eta \leq \frac{\beta-2}{2(2+\beta)}(1+\rho^2),$$

which is the estimate (4.18). Thus the β-admissibility condition for functions in $\Phi_{C,\beta}(\Omega,(1+z)/(1-z))$ is equivalent to $\phi(i\rho,\eta;z) \notin \Omega$, whence $\phi \in \Phi_{C,\beta}(\Omega,(1+z)/(1-z))$. It follows from Theorem 4.4 that $f \in \mathcal{CV}_b$. □

For $\beta = 2$, Definition 4.3 leads to Definition 4.1 in Ali et al. [3, p. 12]. In this case, the coefficient b satisfied the sharp bound for second coefficient when $f \in \mathcal{CV}$, that is, $|b| = |a_2| \leq 1$ and Theorem 4.6 yields the sufficient condition for $f \in \mathcal{A}$ obtained by Ali et al. [3].

Corrollary 4.4. [3, Theorem 4.7, p. 15] *Let Ω be a set in $\mathbb{C}$. Let the function $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfying the admissibility condition*

$$\phi(i\rho,\eta;z) \notin \Omega,$$

for $z \in \mathbb{U}$ and for all real ρ and η with $\eta \leq 0$. Let $f \in \mathcal{A}$ with $f'(z) \neq 0$. If

$$\phi\left(1+\frac{zf''(z)}{f'(z)}, z^2\{f,z\};z\right) \in \Omega, \qquad z \in \mathbb{U},$$

then $f \in \mathcal{CV}$.

If $h(z) = (1+z)/(1-z)$, then clearly $h(\mathbb{U}) = \triangle = q(\mathbb{U})$ and the class of β-admissible functions $\Phi_{C,\beta}(h(\mathbb{U}), \triangle)$ is simply written as $\Phi_{C,\beta}(\triangle)$ and Theorem 4.6 yields the following corollary.

Corollary 4.5. *Let* $\phi \in \Phi_{C,\beta}(\triangle)$ *and satisfy*

$$\operatorname{Re}\left\{\phi(i\rho, \eta; z)\right\} \leq 0,$$

when $\rho, \eta \in \mathbb{R}$ *and* $0 < \beta \leq 2$*, with* $\eta \leq [(\beta-2)(1+\rho^2)]/[2(2+\beta)]$*. If* $f \in \mathcal{A}_b$, $0 \leq b \leq 1$ *with* $f'(z) \neq 0$ *and satisfies*

$$\operatorname{Re}\left\{\phi\left(1+\frac{zf''(z)}{f'(z)},\, z^2\{f,z\}; z\right)\right\} > 0, \qquad z \in \mathbb{U},$$

then $f \in \mathcal{CV}_b$*.*

In the case $\beta = 2$, the coefficient b will follow the same argument as before and Corollary 4.5 reduces to the following corollary.

Corollary 4.6. [35, Theorem 4.6b, p. 246] *Let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *satisfy*

$$\operatorname{Re}\left\{\phi(i\rho, \eta; z)\right\} \leq 0,$$

when $\rho, \eta \in \mathbb{R}$*, and* $\eta \leq 0$*. Let* $f \in \mathcal{A}$ *with* $f'(z) \neq 0$*. If*

$$\operatorname{Re}\left\{\phi\left(1+\frac{zf''(z)}{f'(z)},\, z^2\{f,z\}; z\right)\right\} > 0, \qquad z \in \mathbb{U},$$

then $f \in \mathcal{CV}$*.*

CHAPTER 5

CONCLUSION

This conclusion chapter summarizes the work done in this thesis and gives some possible directions for future research. This thesis utilizes the methodology of differential subordination to study complex-valued analytic functions with fixed initial coefficient as well as with fixed second coefficient. The theory of differential subordination pioneered by Miller and Mocanu that are discussed earlier in the introductory chapter only focused on the general classes of analytic functions. By reformulating the existing theory on differential subordination, the corresponding theory developed by Ali et al. [7] extends to the class of analytic functions with fixed initial coefficient. Some known results of this newly enhanced theory for analytic functions with fixed initial coefficient have been applied to the problems discussed in this thesis.

The linear second-order differential subordination

$$A(z)z^2p''(z)+B(z)zp'(z)+C(z)p(z)+D(z)\prec h(z)$$

plays an important role in the first problem considered. The aim is to determine conditions on the complex-valued functions A,B,C,D and h so that the linear second-order differential subordination will have a β-dominant of the solution. A few special cases of the function h are considered on the linear second-order differential subordination. These special cases of the function h include functions with positive real part, bounded functions and convex functions. The appropriate differential implications corresponding to that particular cases are described geometrically. Under this framework,

β-admissibility condition applied to the complex-valued functions A, B, C and D are obtained to deduce these differential implications. Several earlier known results are obtained as consequences.

As an application, the inclusion properties for linear integral operators on certain subclasses of analytic functions with fixed initial coefficient are investigated in Chapter 3. The linear integral operators are derived from the integral inequalities by using some differential inequalities in Chapter 2. As a consequence, the linear integral operators is shown to map certain subclasses of analytic functions with fixed initial coefficient into itself. These subclasses include class of functions with positive real part, bounded functions and convex functions.

This thesis also examined further the analytic functions with fixed initial coefficient that are associated with the normalized analytic functions with fixed second coefficient as discussed in Chapter 4. An appropriate class of β-admissible functions related to starlikeness and convexity are introduced. Subordination implications involving the Schwarzian derivative are obtained by making use of these classes. As a further application, sufficient conditions in term of Schwarzian derivative for normalized analytic functions with fixed second coefficient to be starlike and convex are obtained. Since the classes introduced by subordination naturally include well-known classes of starlike and convex functions, earlier known results for these classes are consequences of the theorem obtained.

This thesis only deals with first and second-order differential subordination. A possible direction for future research is to investigate higher orders of differential subordination. The definitions for first and second-order differential subordination, presented in the introduction chapter, could be extended very naturally to higher order differential subordination. All of the results dealing with the solution of the third-order differential subordination could be referred in [35, Section 6.2]. By appealing to these results, the linear third-order differential subordination

$$A(z)z^3p'''(z)+B(z)z^2p''(z)+C(z)zp(z)+D(z)p(z)+E(z)\prec h(z),$$

could be considered. The solutions for higher order differential subordination are much more difficult to obtain and very little is known. There is still a great deal of research to be done in this field.

Another possible area for research is radius problems which continue to be an important area of study. The radius problem has been widely studied in recent years. Various authors have investigated several interesting properties of the radius of various classes of functions that are shaped by the coefficients of its mappings.

The radius of a property P in the class $\mathscr{M}$, denoted by $R_P(\mathscr{M})$, is the largest number R such that every function in $\mathscr{M}$ has the property P in the disk $\mathbb{U}_r=\{z\in\mathbb{C}:|z|<r\}$ for every $r<R$. For example, the radius of convexity for the class $\mathcal{S}$ is $2-\sqrt{3}$ since every function $f\in\mathcal{S}$ maps $\mathbb{U}_r$ onto a convex region for $r\le 2-\sqrt{3}$ [22, Theorem 10, p. 119] and the Koebe function k is the form of (1.2) (see Section 1.1) shows that this bound cannot be improved.

Recently, many authors have investigated the problems of finding the radius constant for some subclasses of $\mathcal{A}$. For instance, radius constants for several classes of analytic functions on the unit disk $\mathbb{U}$ which include the radius of starlikeness of a positive order, radius of parabolic starlikeness, radius of Bernoulli lemniscate starlikeness, and radius of uniform convexity can be found in Ali et al. [10]. Further work on the similar problem could be considered for the analytic functions with fixed initial coefficient. Several other subclasses of $\mathcal{A}$ and $\mathcal{S}$ are also of great interest.

REFERENCES

[1] James Waddell Alexander. Functions which map the interior of the unit circle upon simple regions. *The Annals of Mathematics*, 17(1):12–22, 1915.

[2] Rosihan M Ali and V Ravichandran. *Complex Analysis*. Penerbit Universiti Sains Malaysia, Penang, 2007.

[3] Rosihan M Ali, V Ravichandran, and N Seenivasagan. Subordination and superordination on schwarzian derivatives. *J. Inequal. Appl*, 2008:1–18, 2008.

[4] Rosihan M Ali, V Ravichandran, and N Seenivasagan. Subordination and superordination of the liu-srivastava linear operator on meromorphic functions. *Bull. Malays. Math. Sci. Soc.(2)*, 31(2):193–207, 2008.

[5] Rosihan M Ali, V Ravichandran, and N Seenivasagan. Differential subordination and superordination of analytic functions defined by the multiplier transformation. *Math. Inequal. Appl*, 12(1):123–139, 2009.

[6] Rosihan M Ali, V Ravichandran, and N Seenivasagan. Differential subordination and superordination of analytic functions defined by the dziok–srivastava linear operator. *Journal of The Franklin Institute*, 347(9):1762–1781, 2010.

[7] Rosihan M Ali, Sumit Nagpal, and V Ravichandran. Second-order differential subordination for analytic functions with fixed initial coefficient. *Bull. Malays. Math. Sci. Soc.(2)*, 34(3):611–629, 2011.

[8] Rosihan M Ali, Eun Nak Cho, Kumar Naveen Jain, and V Ravichandran. Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination. *Filomat*, 26(3):553–561, 2012.

[9] Rosihan M Ali, Naveen K Jain, and V Ravichandran. Radii of starlikeness associated with the lemniscate of bernoulli and the left-half plane. *Applied Mathematics and Computation*, 218(11):6557–6565, 2012.

[10] Rosihan M Ali, Naveen K Jain, and V Ravichandran. On the radius constants for classes of analytic functions. *Bull. Malays. Math. Sci. Soc.(2)*, 36(1):23–38, 2013.

[11] SD Bernardi. Convex and starlike univalent functions. *Transactions of the American Mathematical Society*, pages 429–446, 1969.

[12] L Bieberbach. Uber die koeffizienten derjenigen potenzreihen, welche eine schlichte abbildung des einheitskreises vermitteln. *Semesterberichte Preussische Akademie der Wissenschaften*, 38:940–955, 1916.

[13] M Biernacki. Sur i'intégrale des fonctions univalentes. *Bulletin de l'Academie Polonaise des Sciences*, 8:29–34, 1960.

[14] Teodor Bulboaca. Integral operators that preserve the subordination. *Bulletin of the Korean Mathematical Society*, 34(4):627–636, 1997.

[15] William M Causey and William L White. Starlikeness of certain functions with integral representations. *Journal of Mathematical Analysis and Applications*, 64 (2):458–466, 1978.

[16] Nak Eun Cho and Oh Sang Kwon. A class of integral operators preserving subordination and superordination. *Bull. Malays. Math. Sci. Soc.(2)*, 33:429–437, 2010.

[17] Martin Chuaqui, Peter Duren, and Brad Osgood. Schwarzian derivative criteria for valence of analytic and harmonic mappings. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 143, pages 473–486. Cambridge Univ Press, 2007.

[18] Martin Chuaqui, Peter Duren, and Brad Osgood. Schwarzian derivatives and uniform local univalence. *Computational Methods and Function Theory*, 8(1): 21–34, 2008.

[19] Louis De Branges. A proof of the bieberbach conjecture. *Acta Mathematica*, 154 (1):137–152, 1985.

[20] PL Duren. *Univalent functions*, volume 259. Grundlehren der Mathematischen Wissenschaften, 1983.

[21] Mark Finkelstein. Growth estimates of convex functions. *Proceedings of the American Mathematical Society*, 18:412–418, 1967.

[22] Adolph Winkler Goodman. *Univalent functions*, volume 1. Mariner Publishing Co., Tampa, Florida, 1983.

[23] Adolph Winkler Goodman. *Univalent functions*, volume 2. Mariner Publishing Co., Tampa, Florida, 1983.

[24] Ian Graham and Gabriela Kohr. *Geometric function theory in one and higher dimensions*. Marcel Dekker Inc., New York, Basel, 2003.

[25] TH Gronwall. On the distortion in conformal mapping when the second coefficient in the mapping function has an assigned value. *Proceedings of the National Academy of Sciences of the United States of America*, 6:300–302, 1920.

[26] Thomas Hakon Gronwall. Some remarks on conformal representation. *The Annals of Mathematics*, 16:72–76, 1914.

[27] J Krzyz and Z Lewandowski. On the integral of univalent functions. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom Physics*, 11(7):447–448, 1963.

[28] Zdzisław Lewandowski, Sanford Miller, and Eligiusz Złotkiewicz. Generating functions for some classes of univalent functions. *Proceedings of the American Mathematical Society*, 56(1):111–117, 1976.

[29] RJ Libera. Some classes of regular univalent functions. *Proceedings of the American Mathematical Society*, 16:755–758, 1965.

[30] Jin-Lin Liu. Certain sufficient conditions for strongly starlike functions associated with an integral operator. *Bull. Malays. Math. Sci. Soc.(2)*, 34(1):21–30, 2011.

[31] AE Livingston. On the radius of univalence of certain analytic functions. *Proceedings of the American Mathematical Society*, 17(2):352–357, 1966.

[32] C Loewner. Untersuchungen über die verzerrung bei konformen abbildungen des einheitsreises $|z| < 1$. die durch funktionen mit nicht verschwindender ableitung geliefert werden. *Ber. Verh. Sächs. Ges. Wiss. Leipzig.*, 69:89–106, 1917.

[33] Sanford S Miller and Petru T Mocanu. Second order differential inequalities in the complex plane. *Journal of Mathematical Analysis and Applications*, 65(2): 289–305, 1978.

[34] Sanford S Miller and Petru T Mocanu. Classes of univalent integral operators. *Journal of Mathematical Analysis and Applications*, 157(1):147–165, 1991.

[35] Sanford S Miller and Petru T Mocanu. *Differential subordinations, Theory and Applications*. Marcel Dekker Inc., New York, Basel, 2000.

[36] Sanford S Miller, Petru Mocanu, and Maxwell O Reade. Starlike integral operators. *Pacific Journal of Mathematics*, 79(1):157–168, 1978.

[37] Sanford S Miller, Petru T Mocanu, et al. Differential subordinations and univalent functions. *The Michigan Mathematical Journal*, 28(2):157–172, 1981.

[38] Sanford S Miller, Petru T Mocanu, and Maxwell O Reade. Subordination-preserving integral operators. *Transactions of the American Mathematical Society*, 283(2):605–615, 1984.

[39] Zeev Nehari. The schwarzian derivative and schlicht functions. *Bulletin of the American Mathematical Society*, 55(6):545–551, 1949.

[40] Zeev Nehari. Some criteria of univalence. *Proceedings of the American Mathematical Society*, 5(5):700–704, 1954.

[41] R Nevanlinna. über die konforme abbildung von sterngebieten. *Översikt av Finska Vetenskaps-Soc. Förh.*, 63(A)(6):1–21, 1921.

[42] Kiyoshi Noshiro. On the theory of schlicht functions. *Journal of the Faculty of Science, Hokkaido Imperial University*, 2(3):129–155, 1935.

[43] Mamoru Nunokawa, Neslihan Uyanik, and Shigeyoshi Owa. Univalency of analytic functions associated with schwarzian derivative. *Acta Universitatis Apulensis Mathematics. Informatic*, (31):157–161, 2012.

[44] Timothy O Opoolaa and Olubunmi A Fadipe-Josepha. The schwarzian derivative

and univalent functions theory. *Sci. Ser. A Math. Sci.(NS)*, 18:57–59, 2009.

[45] Horiana Ovesea-Tudor and Owa Shigeyoshi. An extension of the univalence criteria of nehari and ozaki and nunokawa. *Hokkaido mathematical journal*, 34 (3):533–539, 2005.

[46] Shigeyoshi Owa and Milutin Obradovic. An application of differential subordinations and some criteria for univalency. *Bulletin of the Australian Mathematical Society*, 41(3):487–494, 1990.

[47] Shigeo Ozaki and Mamoru Nunokawa. The schwarzian derivative and univalent functions. *Proceedings of the American Mathematical Society*, 33(2):392–394, 1972.

[48] KS Padmanabhan. Growth estimates for certain classes of convex and close-to-convex functions (the gronwall problem). *J. Indian Math. Soc.(NS)*, 68(1-4): 177–189, 2001.

[49] NN Pascu. Alpha close-to-convex functions. In *Romanian-Finnish Seminar on Complex Analysis*, pages 331–335, 1979.

[50] VV Pokornyi. On some sufficient conditions for univalence. In *Dokl. Akad. Nauk SSSR*, volume 79, pages 743–746, 1951.

[51] V Ravichandran. Certain second order linear differential subordinations. *Journal of Inequalities in Pure and Applied Mathematics*, 5(3), 2004.

[52] V Ravichandran and Sumit Nagpal. Applications of the theory of differential subordination for functions with fixed initial coefficient to univalent functions. *Annales Polonici Mathematici*, 105:225–238, 2012.

[53] Malcolm IS Robertson. On the theory of univalent functions. *Annals of Mathematics*, 37(2):374–408, 1936.

[54] Ram Singh. On bazilevic functions. *Proceedings of the American Mathematical Society*, 38(2):261–271, 1973.

[55] E Strohhäcker. "beiträge zur theorie der schlichten funktionen". *Mathematische Zeitschrift*, 37:356–380, 1933.

[56] David E Tepper. On the radius of convexity and boundary distortion of schlicht functions. *Transactions of the American Mathematical Society*, 150(2):519–528, 1970.

[57] Stefan E Warschawski. On the higher derivatives at the boundary in conformal mapping. *Transactions of the American Mathematical Society*, 38(2):310–340, 1935.

[58] Ri-Guang Xiang, Zhi-Gang Wang, and Maslina Darus. A family of integral operators preserving subordination and superordination. *Bull. Malays. Math. Sci. Soc.(2)*, 33(1):121–131, 2010.